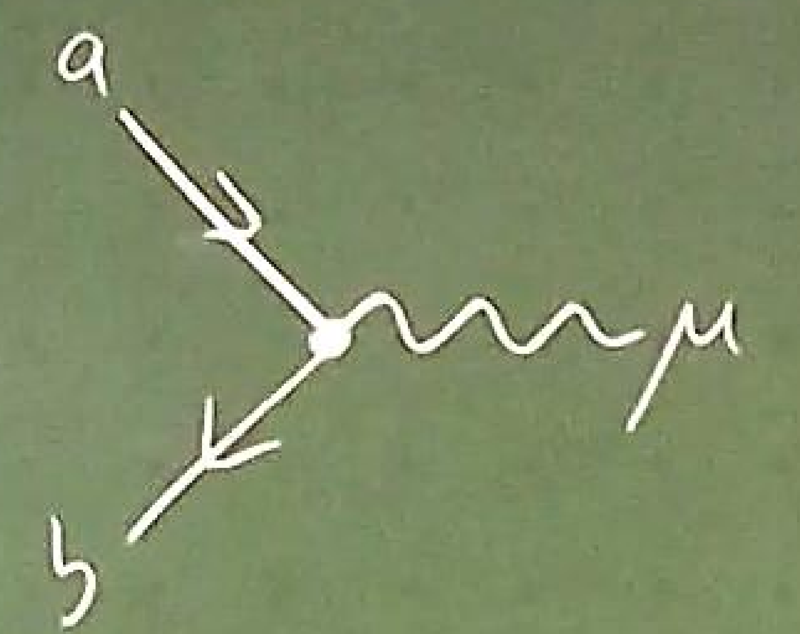
Momentum space Feynman rules for scattering amplitudes in QED

Propagators:

$$a \xrightarrow{\;p\;} b = \frac{i(\not{p}+m)_{ba}}{p^2-m^2+i\varepsilon}$$

$$\mu \overset{q}{\rightsquigarrow} \nu = \frac{-ig_{\mu\nu}}{q^2+i\varepsilon}$$

Vertices:



$$= -ie\gamma^{\mu}_{ba}$$

External legs:

Fermion

$$a \longleftarrow| s = u^s_a(p) \quad \text{in}$$

$$s |\longleftarrow a = \bar{u}^s_a(p) \quad \text{out}$$

Anti-fermion

$$a \longrightarrow| s = \bar{v}^s_a(p) \quad \text{in}$$

$$s |\longrightarrow a = v^s_a(p) \quad \text{out}$$

Photon

$$\mu \rightsquigarrow | r = \varepsilon^r_\mu(q) \quad \text{in}$$

$$r | \rightsquigarrow \mu = \varepsilon^{r*}_\mu(q) \quad \text{out}$$

Evaluation:

1. Impose momentum cons. at vertices
2. Integrate undetermined momenta
3. Compute overall sign of the diagram

Minimal coupling

= only couple to first moments of charge distribution (= charges)

$$\mathcal{L} = \bar{\psi}(i\not{\partial}-m)\psi - \frac{1}{4}F^2$$

$$-e\underbrace{\bar{\psi}\gamma^\mu\psi}_{\text{charge current}} A_\mu \quad \Big\}\ \text{minimal coupling}$$

$$\underbrace{-a}_{\frac{eK}{8m}}\ \bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu} \quad \Big\}\ \text{Pauli coupling}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$$

$$\Rightarrow (i\not{D} - m - a\sigma^{\mu\nu}F_{\mu\nu})\psi = 0$$

$$\Rightarrow \vec{\mu}_{\text{eff}} = \frac{e}{2m}\underbrace{(2+K)}_{g}\vec{S}$$

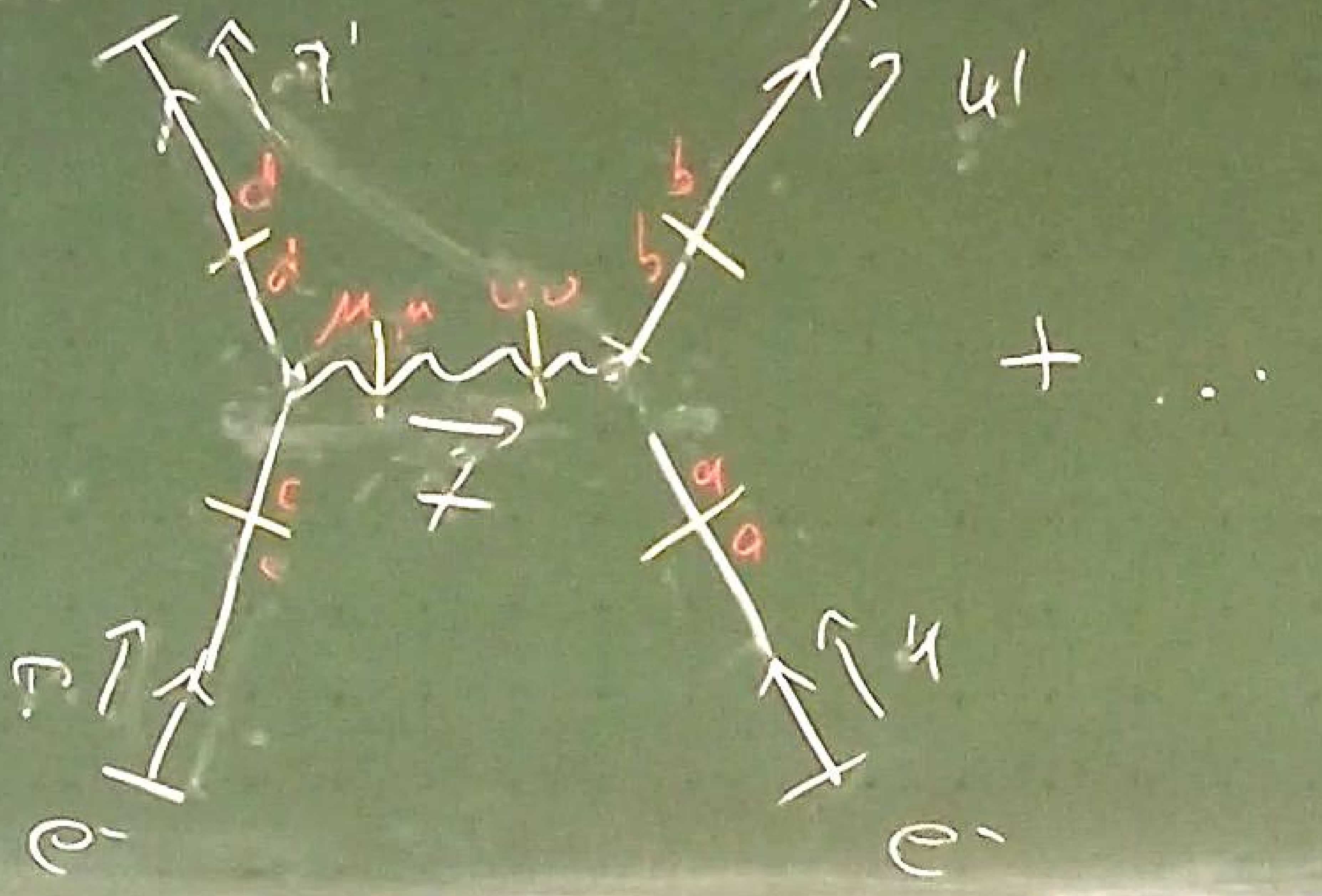
The Coulomb potential

1) **Møller scattering.**

$$e^- + e^- \longrightarrow e^- + e^-$$

i) Contribution to tree-level amplitude:

$$i\mathcal{M}\left(e^-(p)\,e^-(k) \longrightarrow e^-(p')\,e^-(k')\right)$$

$$=$$  $$+ \dots$$

$$= \sigma \cdot \bar{u}_d(p')(-ie\gamma^\mu_{dc})u_c(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}_b(k')(-ie\gamma^\nu_{ba})u_a(k)$$

$$= \sigma\; \bar{u}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k')(-ie\gamma^\nu)u(k)$$

$$p - p' = q = k' - k$$

ii) **Nonrelativistic limit.** $|\vec{p}|^2 \ll m^2$

- $u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$
- $\dfrac{1}{(p-p')^2} \approx \dfrac{-1}{|\vec{p}-\vec{p}\,'|^2}$

$$\rightarrow \bar{u}(p')\gamma^\mu u(p) \approx \begin{cases} 2m\,\xi^\dagger_{p'}\xi_p & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases}$$

$$\rightarrow i\mathcal{M} = \boxed{\sigma\,\frac{-ie^2}{|\vec{p}-\vec{p}\,'|^2}}\,\left(2m\,\xi^\dagger_{p'}\xi_p\right)\left(2m\,\xi^\dagger_{k'}\xi_k\right)$$

iii) Compare with non-relativistic scattering theory (first Born approxim.)

$$\langle \vec{p}\,'|iT|\vec{p}\rangle \overset{*}{=} \boxed{-i\hat{V}(\vec{q})}(2\pi)\,\delta(E_{\vec{p}'} - E_{\vec{p}})$$

$$\vec{q} = (\vec{p}\,' - \vec{p})$$

Fourier transform of scattering potential

$$\rightarrow \hat{V}(\vec{q}) = \sigma\,\frac{e^2}{|\vec{q}|^2} \Rightarrow \boxed{V(\vec{r}) = \sigma\,\frac{\overset{\alpha}{e^2}}{4\pi|\vec{r}|}}$$

iv) Sign:

$$ {}_{aa}\langle p', k'| \, \bar\Psi \Psi A \, \bar\Psi \Psi A \, |p, k\rangle_{aa} $$

$$ = \langle 0| a_{k'} a_{p'} \, \bar\Psi \Psi A \, \bar\Psi \Psi A \, a_p^\dagger a_k^\dagger |0\rangle $$

$\rightarrow \quad 1 + 1 + 2 = 4 \quad \rightarrow (-1)^{\#} = 1$

$\sigma = +1$

v)

$$ V_{e^-e^-}(r) = + \frac{e^2}{4\pi r} $$

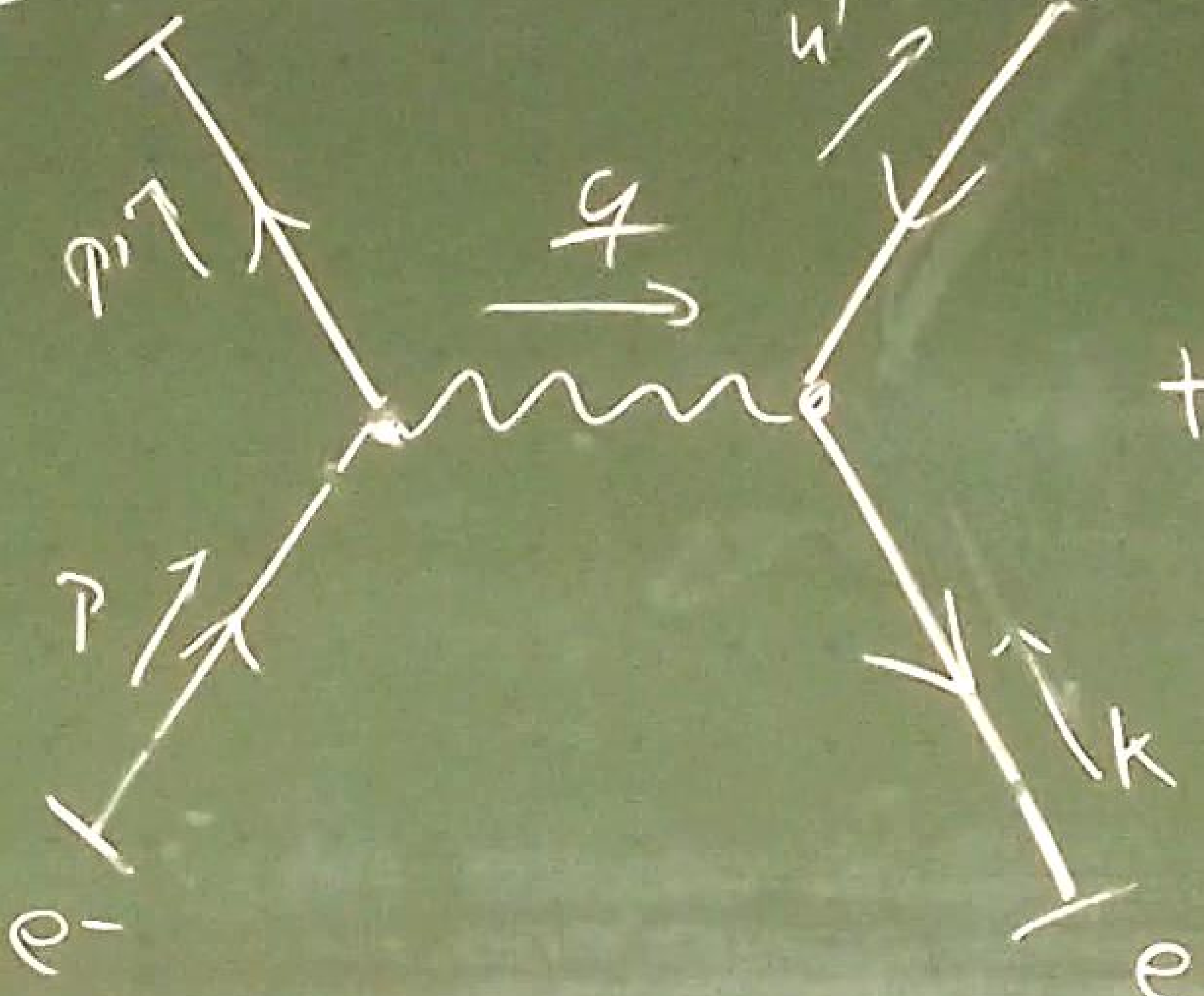
$\rightarrow$ Equal charges repel each other ✓

2) Bhabha scattering

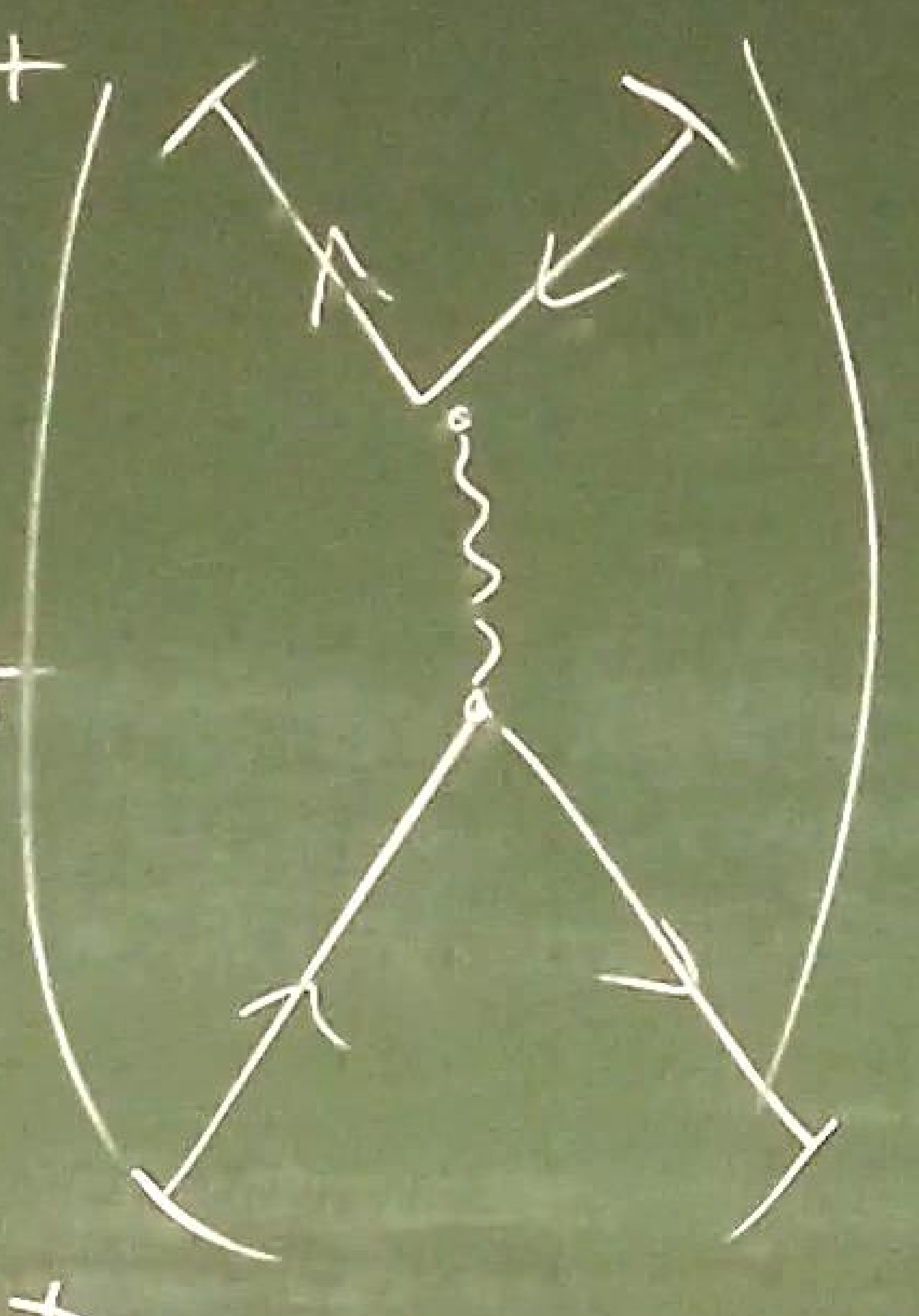
$$ e^- + e^+ \longrightarrow e^- + e^+ $$

i) iM

$=$



$+$



$$ = \sigma \, \bar u(p')(-i\gamma^\mu) u(p) \left(\frac{-i g_{\mu\nu}}{q^2} \right) \bar v(k)(-i\gamma^\nu) v(k') $$

$$ p - p' = q = k' - k $$

ii) Non-rel limit $\rightarrow$ Same result with $k \leftrightarrow k'$

but now is σ?

iii) $\langle 0| b_{k'} a_{p'} \, \bar\Psi \Psi A \, \bar\Psi \Psi A \, a_p^\dagger b_k^\dagger |0\rangle$

$\rightarrow \quad 2 + 1 + 2 = 5$

$\rightarrow \quad (-1)^{\#} = -1 \Rightarrow \sigma = -1$

iv) $\boxed{V_{e^+e^-}(r) = - \frac{e^2}{4\pi r}}$ $\rightarrow$ Attractive Coulomb potential

5. Elemantary Processes of QED

5.1. Cross section $e^+e^- \to \mu^+\mu^-$ scattering

Reaction:

1] $e^- + e^+ \longrightarrow \mu^+ + \mu^-$

$m_e \ll m_\mu$

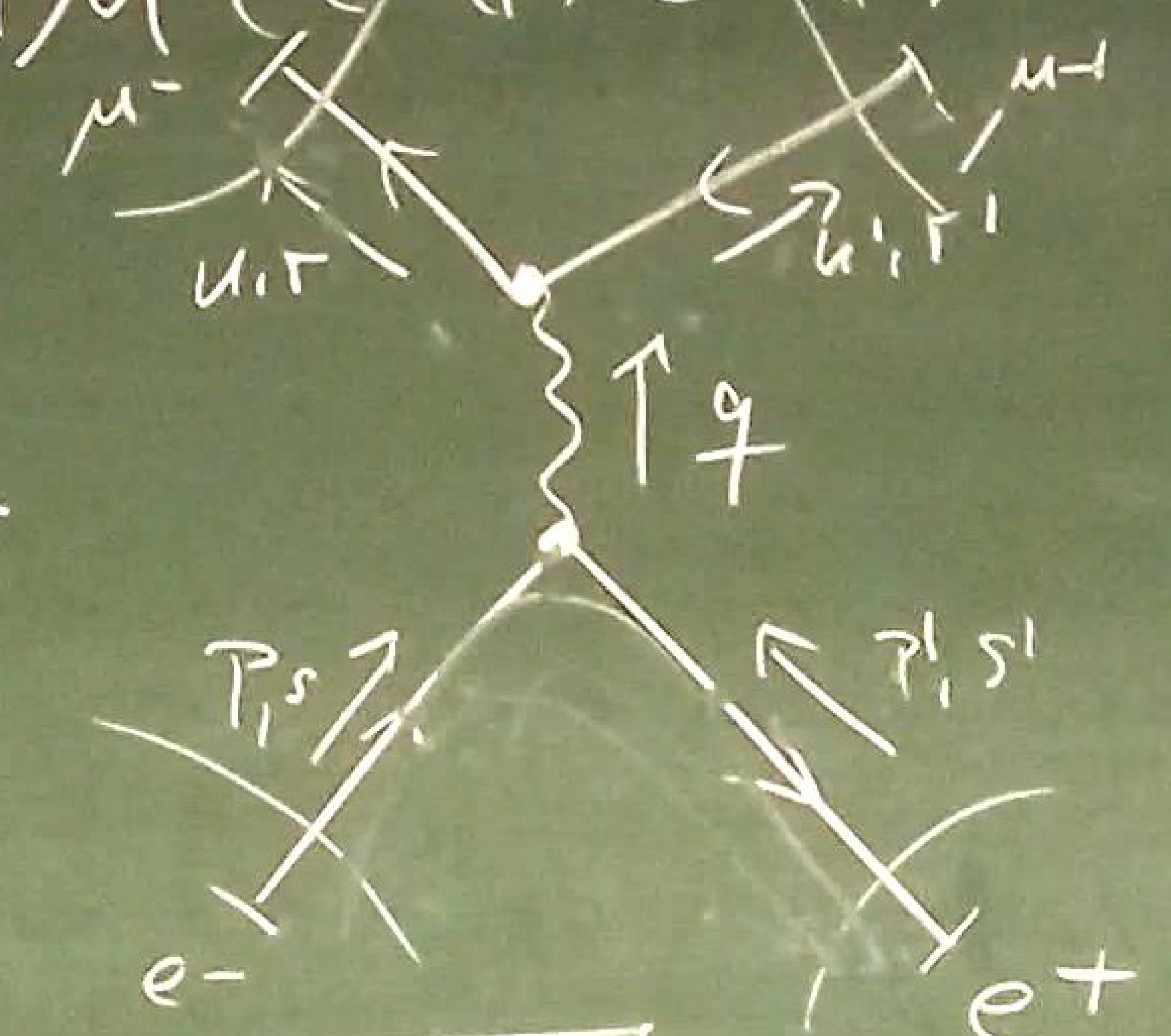
2] Note: $q_e = q_\mu = e = -|e|$

$$\mathcal{L}^{e,\mu} = \sum_{f=e,\mu} \bar\psi_f (i\not\partial - m_f)\psi_f - q_f \bar\psi_f \gamma^\mu \psi_f A_\mu - \frac{1}{4}F^2$$

3] Tree-level amplitude:

$$i\mathcal{M}(e^-(p)\, e^+(p') \longrightarrow \mu^-(k)\, \mu^+(k'))$$

$=$



$$= \underbrace{\bar v_e^{s'}(p')(-i q_e \gamma^\mu) u_e^s(p)}_{\text{electron}} \left(\frac{-i g_{\mu\nu}}{q^2}\right) \underbrace{\bar u_\mu^r(k)(-i q_\mu \gamma^\nu) v_\mu^{r'}(k')}_{\text{muon}}$$

$$= \frac{i e^2}{q^2} \left(\bar v(p')\gamma^\mu u(p)\right)\left(\bar u(k)\gamma_\mu v(k')\right),$$

$p + p' = q = k + k'$

4.] $d\sigma \propto |\mathcal{M}|^2 \to \mathcal{M}^*$

Use $(\bar v \gamma^\mu u)^* \stackrel{\circ}{=} \bar u \gamma^\mu v$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \boxed{\left[(\bar v(p')\gamma^\mu u(p))(\bar u(p)\gamma^\nu v(p'))\right]} \cdot \left[(\bar u(k)\gamma_\mu v(k'))(\bar v(k')\gamma_\nu u(k))\right]$$

5. Typical collider setup:

- e^+, e^- beams unpolarized
  → average over spins of in-state
- Muon detector cannot resolve spin
  → Sum over spins of out-state

→ $d\sigma \propto \frac{1}{4} \sum_{ss'} \sum_{rr'} |\mathcal{M}(ss' \to rr')|^2$

6. Use spin sum relations.

$$\sum_s u^s \bar{u}^s = \not{p} + m \qquad v \bar{v}^s = \not{p} - m$$

$$\sum_{ss'} \left(\bar{v}^{s'}_a(p') \gamma^\mu_{ab} u^s_b(p)\right)\left(\bar{u}^s_c(p) \gamma^\nu_{cd} v^{s'}_d(p')\right)$$

$$\overset{!}{=} \mathrm{Tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]$$

7. →

$$\frac{1}{4} \sum_{ss'rr'} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \mathrm{Tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] \mathrm{Tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right]$$

8. Trace technology

Examples: $\mathrm{Tr}[\text{odd } \gamma^\mu] = 0$

- $\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$

  ⋮

- $\gamma^\mu \gamma_\mu = 4 \mathbb{1}$

  $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$

9) $\longrightarrow$

$\cdot\ \mathrm{Tr}\left[(\not p' - m_e)\gamma^\mu(\not p + m_e)\gamma^\nu\right]$

$\overset{\circ}{=} 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(pp' + m_e^2)\right]$

10) $m_e/m_\mu \approx 1/200 \longrightarrow m_e = 0$

$$\underbrace{\frac{1}{4}\sum_{ss'rr'} |M|^2}_{\overline{|M|^2}} \overset{\circ}{=} \frac{8e^4}{q^4}\left[(pu)(p'u') + (pu')(p'u) + m_\mu^2(pp')\right]$$

11) Centre of mass frame: $\vec p + \vec p\,' = 0 = \vec u + \vec u\,'$

- $p = (E, E\hat z)$, $p' = (E, -E\hat z)$
- $|\vec u| = \sqrt{E^2 - m_\mu^2}$
- $\vec u \cdot \hat z = |\vec u| \cos\theta$

$\Downarrow$

- $q^2 = (p+p')^2 = 4E^2$
- $pp' = 2E^2$
- $pu = p'u' = E^2 - E|\vec u|\cos\theta$

$pu' = p'u = E^2 + E|\vec u|\cos\theta \Rightarrow \boxed{\overline{|M|^2} = e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]}$