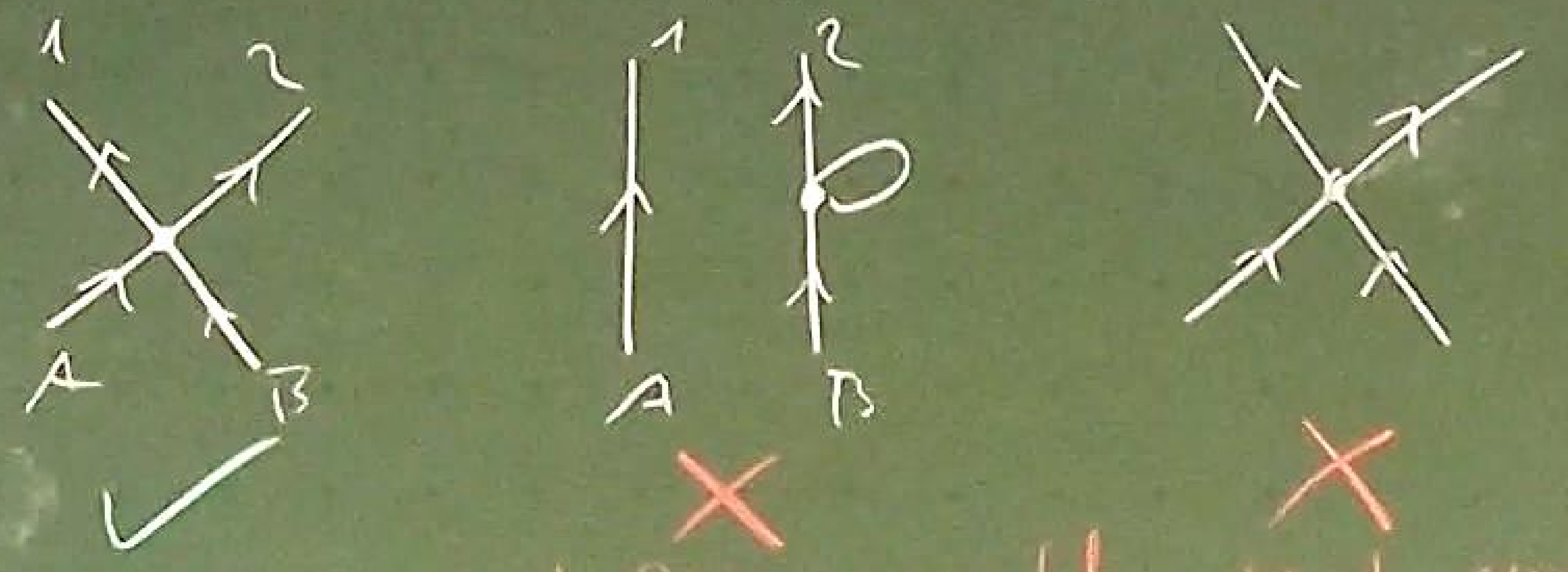
Recap

$$\langle \vec{p}_1 \ldots \vec{p}_n | iT | \vec{p}_A \vec{p}_B \rangle = i\mathcal{M}(p_A p_B \to \{p_f\})\,(2\pi)^4 \delta^{(4)}(p_A + p_B - \textstyle\sum p_f)$$

$$= \sum \left\{ \begin{array}{l} \underline{\text{fully connected}} + \underline{\text{amputated}} \\ \text{Feynman diagrams with} \\ \vec{p}_A, \vec{p}_B \text{ incoming} \,/\, \{p_f\} \text{ outgoing} \end{array} \right\}$$



Feynman rules in

Position space

1. Edges: $x \underline{\qquad} y = D_F(x-y)$
2. Internal vertices:

$= (-i\lambda) \int d^4 z$

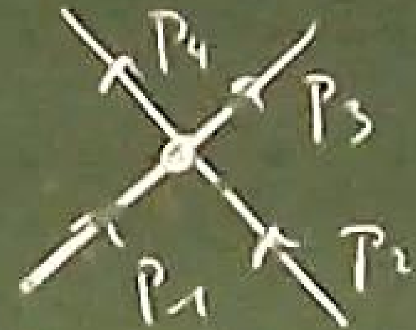
3. External lines:

$= e^{-ipz}$

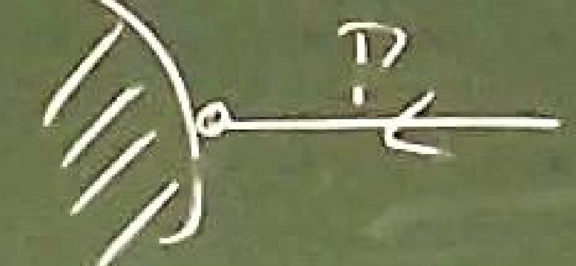
4. —
5. Divide by sym. factor $\frac{1}{S}$

momentum space

$\xrightarrow{p} = \frac{i}{p^2 - m^2 + i\varepsilon}$



$= (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$



$= 1$

Integrate internal momenta: $\prod \int \frac{d^4 p_i}{(2\pi)^4}$

—, —

4.7. Feynman Rules for QED

Setting the Stage

1) Fields:

Fermions:	$\psi(x)$	(bispinor field)
Photons:	$A_\mu(x)$	(vector field)

2) Lagrangian:

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int}$$

$$= \bar\psi(i\partial\!\!\!/ - m)\psi - \frac{1}{4}\underbrace{F_{\mu\nu}}_{\partial_\mu A_\nu - \partial_\nu A_\mu}F^{\mu\nu} - \boxed{e}\underbrace{\bar\psi\gamma^\mu\psi}_{j^\mu}A_\mu$$

mass of fermion

charge of fermion

$$= \bar\psi(i\underbrace{D\!\!\!\!/}_{\text{covariant derivative}} - \boxed{m})\psi - \frac{1}{4}F^2$$

$D_\mu = \partial_\mu + ieA_\mu$

3) Hamiltonian:

$$H_{QED} = H_{Dirac} + H_{Maxwell} + \underbrace{H_{int}}_{e\int d^3x\, \bar\psi\gamma^\mu\psi A_\mu}$$

4) EOM:

$$(iD\!\!\!\!/ - m)\psi = 0$$

$$\partial_\nu F^{\nu\mu} = j^\mu$$

Note 4.2 $\mathcal{L}_{QED}$ is invariant under local $U(1)$ gauge transformations.

$$\psi'(x) = e^{ie\alpha(x)}\psi(x)$$

$$A'(x) = A_\mu(x) - \partial_\mu\alpha(x)$$

for arbitrary $\alpha: \mathbb{R}^{1,3} \to \mathbb{R}$.

Note 43 Standard model:

$$\text{"}\mathcal{L}^{\text{SM}}_{\text{QED}}\text{"} = \sum_f \left\{ \bar{\Psi}_f (i\not{\partial} - m)\Psi_f - \underbrace{q_f \bar{\Psi}_f \gamma^\mu \psi}_{J_f^\mu} A_\mu \right\} - \frac{1}{4}F^2$$

$f \in \{\text{Leptons, Quarks}\}$

$= \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, c, s, t, b\}$

Notes on Fermion / Dirac sector

<u>Remember</u>: $$S_F^{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip(x-y)}$$

$$= \begin{cases} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle & x^0>y^0 \\ -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle & y^0>x^0 \end{cases}$$

$$= \langle 0|T\{\psi_a(x)\bar{\psi}_b(y)\}|0\rangle$$

$\to$ need Wick's theorem

<u>1] Time-ordering:</u> $\psi \in \{\psi, \bar{\psi}\}$

$$T\{\psi_{\sigma_1} \ldots \psi_{\sigma_N}\} \equiv (-1)^{\#} \psi_1 \ldots \psi_N \qquad x_1^0 > \ldots > x_N^0$$

$(-1)^{\#}$: Signum of permutation σ

<u>2] Normal order.</u> $X \in \{a_p^s, a_p^{s\dagger}, b_p^s, b_p^{s\dagger}\}$

$$:X_1 \ldots X_N: \equiv (-1)^{\#} (\text{creation operator}) \times (\text{annihilation op.})$$

number of operator interchanges

<u>3] Contractions:</u>

$$\overbracket{\psi_a(x)\psi_b(y)} \equiv T\{\psi_a(x)\psi_b(y)\} - :\psi_a(x)\psi_b(y):$$

$$\to \overbracket{\psi_a(x)\bar{\psi}_b(y)} \doteq \begin{cases} \{\psi_a^+(x), \bar{\psi}_b^-(y)\} & x^0>y^0 \\ -\{\bar{\psi}_b^+(y), \psi_a^-(x)\} & x^0<y^0 \end{cases} = S_F^{ab}(x-y)$$

$$\overbracket{\psi_a(x)\psi_b(y)} \doteq 0$$

$$\overbracket{\bar{\psi}_a(x)\bar{\psi}_b(y)} \doteq 0$$

4] Contraction + Normal order:

$$:A\,\overbrace{\psi_a(x)\,B\,\psi_b}(y)\,C: \;\equiv (-1)^{\#}\,\overbrace{\psi_a(x)\psi_b(y)}\;:ABC:$$

# of operator interchanges (ψ_a with A, ψ_b with AB)

5] Wick's theorem

$$\boxed{T\{\psi_a(x_1)\psi_b(x_2)\ldots\} = :\psi_a(x_1)\psi_b(x_2)\ldots + \text{all possible contractions}:}$$

Notes on Photon/Maxwell Scalar

1] Observation: A^μ: four dof of freedom but photon has only 2 dof.

2] Problem: Gauge invariance
→ Unphysical dof
→ Fix gauge to quantize only phys dof

3] Different solutions:
- Coulomb gauge: $\nabla\vec{A}=0$
- Lorentz gauge $\partial_\mu A^\mu=0$ is Lorentz invariant ⇒ Gupta-Bleuler quantization
- Faddeev-Popov procedure (↪ later)

4] Motivate:
i) $\mathcal{L}$ $\partial_\mu A^\mu=0$ $\xrightarrow{\text{EOM}}$ $\underbrace{\partial^2}_{\Box} A^\mu=0$

ii) Expand field in classical solutions.

$$A_\mu(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{r=0}^{3}\Big[\underbrace{a_p^r\,\boxed{\varepsilon_\mu^r(p)}}\,e^{-ipx} + a_p^{r\dagger}\,\varepsilon_\mu^{r*}(p)\,e^{ipx}\Big]$$

polarization vectors

$p^0=|\vec{p}|$

$\boxed{p^2=0}$

5] Result:
i) For physical (external) photons:

$$\boxed{\varepsilon^\mu(p)=\begin{pmatrix}0\\ \vec{\varepsilon}(p)\end{pmatrix},\quad \vec{\varepsilon}(p)\cdot\vec{p}=0}$$

transverse pol

→ Two $r,s = 1,2$ independent bosonic modes for each momentum $\vec{p}$:

$$\boxed{[a_{p}^{r\mu}, a_{q}^{s\nu\dagger}] = (2\pi)^3\, \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q})}$$

$-g^{\mu\nu} = -1$ (+), (−) $+1$

$\mu,\nu = 0,1,2,3$

ii) Propagator

$$\boxed{\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle \overset{*}{=} \int \frac{d^4q}{(2\pi)^4}\, \frac{-i\, g_{\mu\nu}}{q^2+i\varepsilon}\, e^{-iq(x-y)}}$$

Feynman rules

1. Expectations: $S_F^{ba} = \psi_b \bar{\psi}_a$

a) Fermions: $a \longrightarrow b$

Photons: $\mu \sim\sim\sim\sim \nu$ $\quad A_\mu(x) A_\nu(y)$

→ Two particle types
(anti)fermions / photons

→ Fermions/Antifermions: $|\vec{p}, s\rangle_{a,b}$ spin ↑↓

Photon $|\vec{p}, \xi\rangle$ polarization

b) Interaction: $(\bar{\psi}_b\, \gamma^\mu_{ba}\, \psi_a) A$

→ Vertex:

2) Momentum-space Feynman rules

- $\longrightarrow\bullet$ internal vertex
- $\vdash$ external 'vertex' $(\,.\,)$
- — virtual cut

$\gamma^\mu . \psi \qquad \bar{\psi}\gamma^\mu \psi A_\mu$

$(\ldots.)(\,.\,)^\mu$

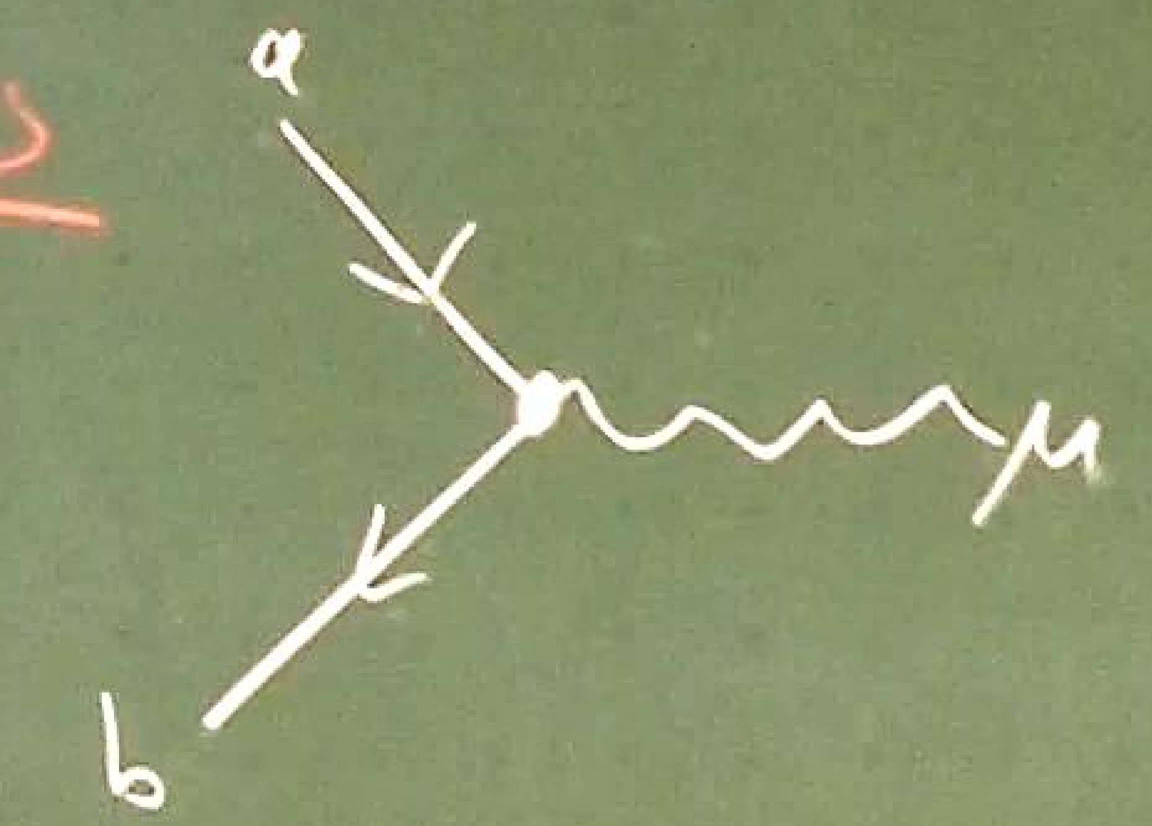
Propagators

Fermions: $a \xrightarrow{p} b = \dfrac{i(\not{p}+m)_{ba}}{p^2-m^2+i\varepsilon} \;\hat{=}\; \psi(x)_b\,\bar{\psi}(y)_a$

$a \longrightarrow b$ simplified

Photons: $\mu \overset{q}{\sim\sim\sim} \nu = \dfrac{-ig_{\mu\nu}}{q^2+i\varepsilon} \;\hat{=}\; A_\mu(x)A_\nu(y)$

Vertices



$= -ie\,\gamma^\mu_{ba} \;\hat{=}\; -ie\int d^4z\,\gamma^\mu_{ba}$

External legs:

Fermions:

$c \longleftarrow| s$ (p) $= u^s_c(p) \;\hat{=}\; \psi_c|p,s\rangle_a$

$s |\longleftarrow c$ (p) $= \bar{u}^s_c(p) \;\hat{=}\; {}_a\langle p,s|\bar{\psi}_c$

Antifermions:

$c \longrightarrow| s$ (p) $= \bar{v}^s_c(p) \;\hat{=}\; \bar{\psi}_c|p,s\rangle_b$

$s |\longrightarrow c$ (p) $= v^s_c(p) \;\hat{=}\; {}_b\langle p,s|\psi_c$

Photons:

$\mu \sim\sim\sim| r$ (q) $= \varepsilon^r_\mu(q) \;\hat{=}\; A_\mu|q,r\rangle$

$|\sim\sim\sim \mu$ (q) $= \varepsilon^{r*}_\mu(q) \;\hat{=}\; \langle q,r|A_\mu$

Evaluate:

1. Impose momentum conservation at internal vertices
2. Integrate over all undetermined momenta
3. Compute the sign of the diagram