Recap

S-matrix:

$$\langle \underbrace{\vec{p}_1 \ldots \vec{p}_n}_{\substack{\text{out}\\ T\to+\infty}} | \underbrace{\vec{k}_A \vec{k}_B}_{T\to-\infty} \rangle_{in} = \lim_{T\to\infty} {}_{t_0}\langle \vec{p}_1 \ldots \vec{p}_n | e^{-iH(2T)} | \vec{k}_A \vec{k}_B \rangle_{t_0}$$

$$\equiv {}_{t_0}\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{k}_A \vec{k}_B \rangle_{t_0}$$

T-matrix:

$$S = \mathbb{1} + \underbrace{iT}_{\substack{\text{non-trivial}\\ \text{scattering}}}$$

Invariant matrix element:

$$\langle \vec{p}_1 \ldots \vec{p}_f | iT | \vec{k}_A \vec{k}_B \rangle = \underbrace{(2\pi)^4 \delta^{(4)}\left(k_A + k_B - \sum p_f\right)}_{\text{4-momentum conservation}} \cdot i \underset{?}{\boxed{\mathcal{M}\left(k_A k_B \to \{p_f\}\right)}}$$

Scattering cross section:

$$\boxed{d\sigma = \underbrace{\frac{1}{2E_{\vec{p}_A} 2E_{\vec{p}_B} |v_A - v_B|}}_{\text{not LI!}} \underbrace{\left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{\vec{p}_f}}\right)}_{LI} \underbrace{\left|\mathcal{M}\left(p_A p_B \to \{p_f\}\right)\right|^2}_{LI \;\; (\to \text{Weinberg})} \underbrace{(2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right)}_{LI}}$$

2 outgoing particles, $m_A = m_B = m_1 = m_2$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\left|\mathcal{M}\left(p_A p_B \to p_1 p_2\right)\right|^2}{64\pi^2 E_{cm}^2}}$$

$$E_{cm} = \left[E_{\vec{p}_A} + E_{\vec{p}_B}\right]_{cm} = \sqrt{(p_A + p_B)^2}$$

$$\vec{p}_A + \vec{p}_B = 0$$

4.6 Computing S-matrix elements from Feynman diagrams

Motivation

We want

1) $\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{p}_A \vec{p}_B \rangle = \lim_{T\to\infty} \langle \vec{p}_1 \ldots \vec{p}_n | e^{-iH2T} | \vec{p}_A \vec{p}_B \rangle$

Eigenstates of H_0

2) Problem

- $|p_A p_B\rangle_0 = \sqrt{2E_A}\sqrt{2E_B}\, a^\dagger_{p_A} a^\dagger_{p_B} |0\rangle$
- $|p_A p_B\rangle = \,?$ $\qquad |\Omega\rangle$: Eigenstate of H

3) Remember

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

4) Assume it holds

$$|p_A p_B\rangle = \lim_{T\to\infty(1-i\varepsilon)} (??)\, e^{-iHT}|p_A p_B\rangle_0$$

$$\lim_{T\to\infty} e^{-iH2(T-i\varepsilon)}$$

5) If this holds:

$$\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{p}_A \vec{p}_B \rangle \propto \langle \vec{p}_1 \ldots | \left(e^{-iHT(1-i\varepsilon)}\right)^\dagger e^{iH2T} e^{-iH(T-i\varepsilon)} |p_A p_B\rangle_0$$

$$\propto \lim_{T\to\infty(1-i\varepsilon)} \langle p_1 \ldots | e^{-iH2T} | p_A p_B\rangle_0$$

$$U(T,-T) = T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] = e^{iH_0(T-t_0)}\, e^{-iH2T}\, e^{-iH_0(T-t_0)}$$

$$\propto \lim_{T\to\infty(1-i\varepsilon)} \langle p_1 \ldots | T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] |p_A p_B\rangle_0$$

6) Correct result:

$$\boxed{\langle \vec{p}_1 \ldots \vec{p}_n | iT | \vec{p}_A \vec{p}_B \rangle = \lim_{T\to\infty(1-i\varepsilon)} \left\{ \langle \vec{p}_1 \ldots \vec{p}_n | T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] |p_A p_B\rangle_0 \right\}_{fc+a}}$$

$fc+a$ "fully connected + amputated"

(LSZ reduction formula Proof: P&S 7.2)

Interpretation + Application

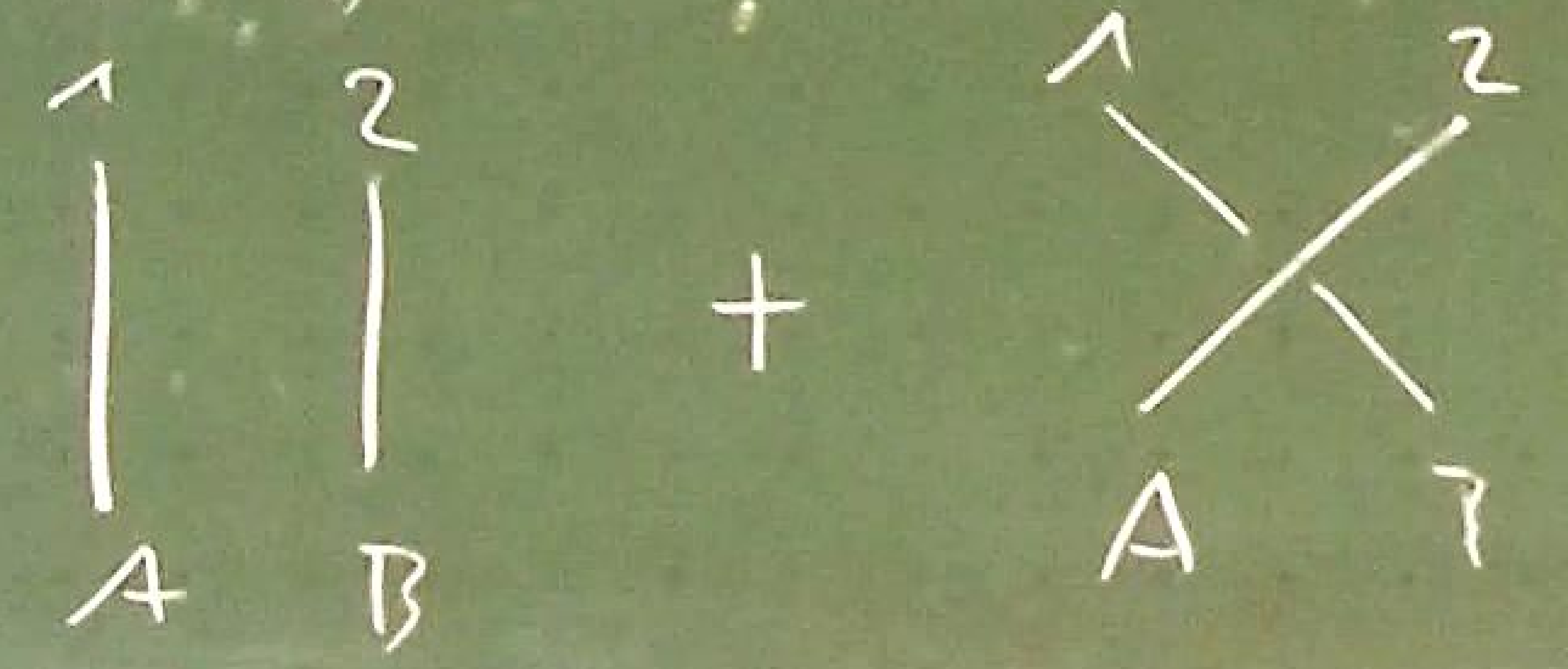
X ϕ^4-theory

1] λ^0-order

$${}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle_0 = \sqrt{2E_{p_1} 2E_{p_2} 2E_{p_A} 2E_{p_B}}\ \langle 0 | a_{p_1} a_{p_2} a^\dagger_{p_A} a^\dagger_{p_B} | 0 \rangle$$

$$\overset{\circ}{=} 2E_{p_A} 2E_{p_B} (2\pi)^6 \left\{ \delta^{(3)}(\vec{p}_1 - \vec{p}_A)\, \delta^{(3)}(\vec{p}_B - \vec{p}_2) + \delta^{(3)}(\vec{p}_A - \vec{p}_2)\, \delta^{(3)}(\vec{p}_B - \vec{p}_1) \right\} \quad (*)$$

$$= \qquad + $$



→ States do not change
→ Contributes $\mathbb{1}$ in $S = \mathbb{1} + iT$

2] λ^1 order:

i] ${}_0\langle \vec{p}_1 \vec{p}_2 | \; -i\frac{\lambda}{4!} \int d^4x \; T(\phi_I^4(x)) \; | p_A p_B \rangle_0$

Wick's theorem

$= \quad : \phi_I^4(x) + \text{contractions} :$

ii] Careful:

$$\phi_I^+(x) | p \rangle_0 = \int \ldots \; a_q \; a_p^\dagger | 0 \rangle \overset{\circ}{=} e^{-ipx} | 0 \rangle$$

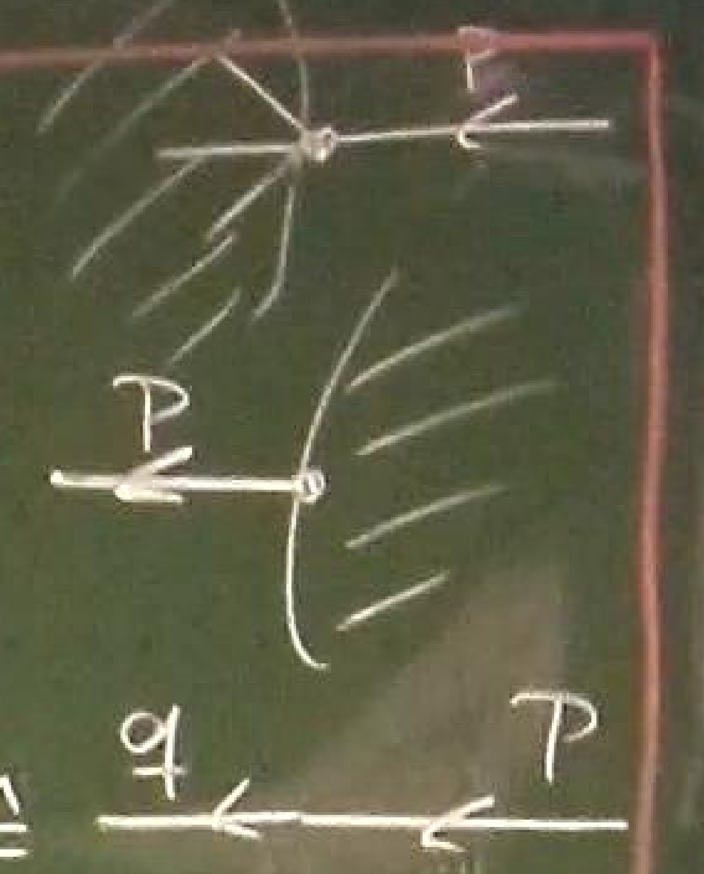
$${}_0\langle p | \phi_I^-(x) = \ldots \ldots = \langle 0 | e^{+ipx}$$

iii] Definitions:

$$\phi_I(x) | \vec{p} \rangle \equiv e^{-ipx} | 0 \rangle \;\hat{=}$$

$$\langle \vec{p} | \phi_I(x) \equiv \langle 0 | e^{+ipx} \;\hat{=}$$

$$\langle \vec{p} | \vec{q} \rangle \equiv 2E_p (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \;\hat{=}$$



iv]

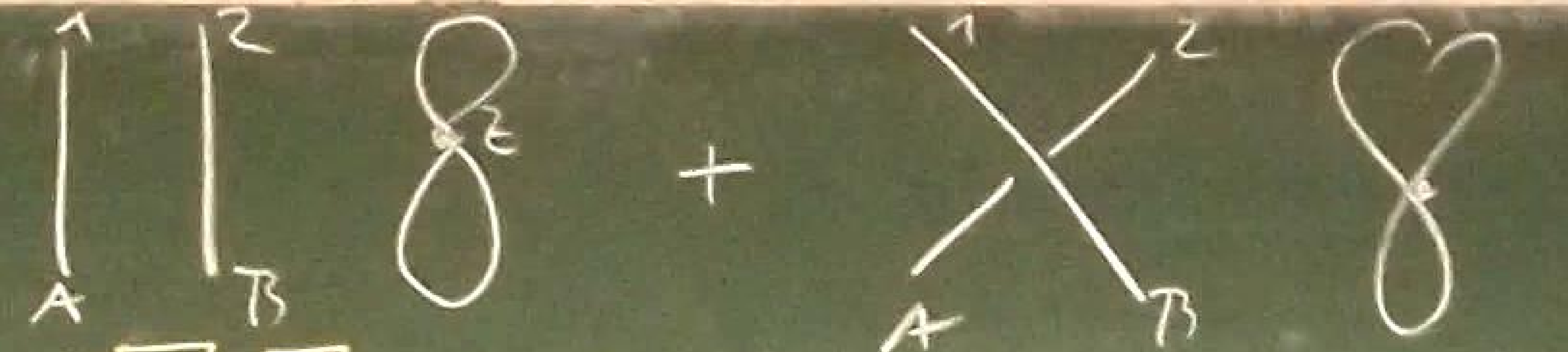
$${}_0\langle \vec{p}_1 \ldots | T\{\phi_a \ldots \phi\} | p_A \ldots \rangle_0$$

$$= \left\{ \begin{array}{l} \text{Sum of all } \underline{\text{full}} \text{ contractions} \\ \text{of fields } \underline{\text{and}} \text{ external-state momenta} \end{array} \right\}$$

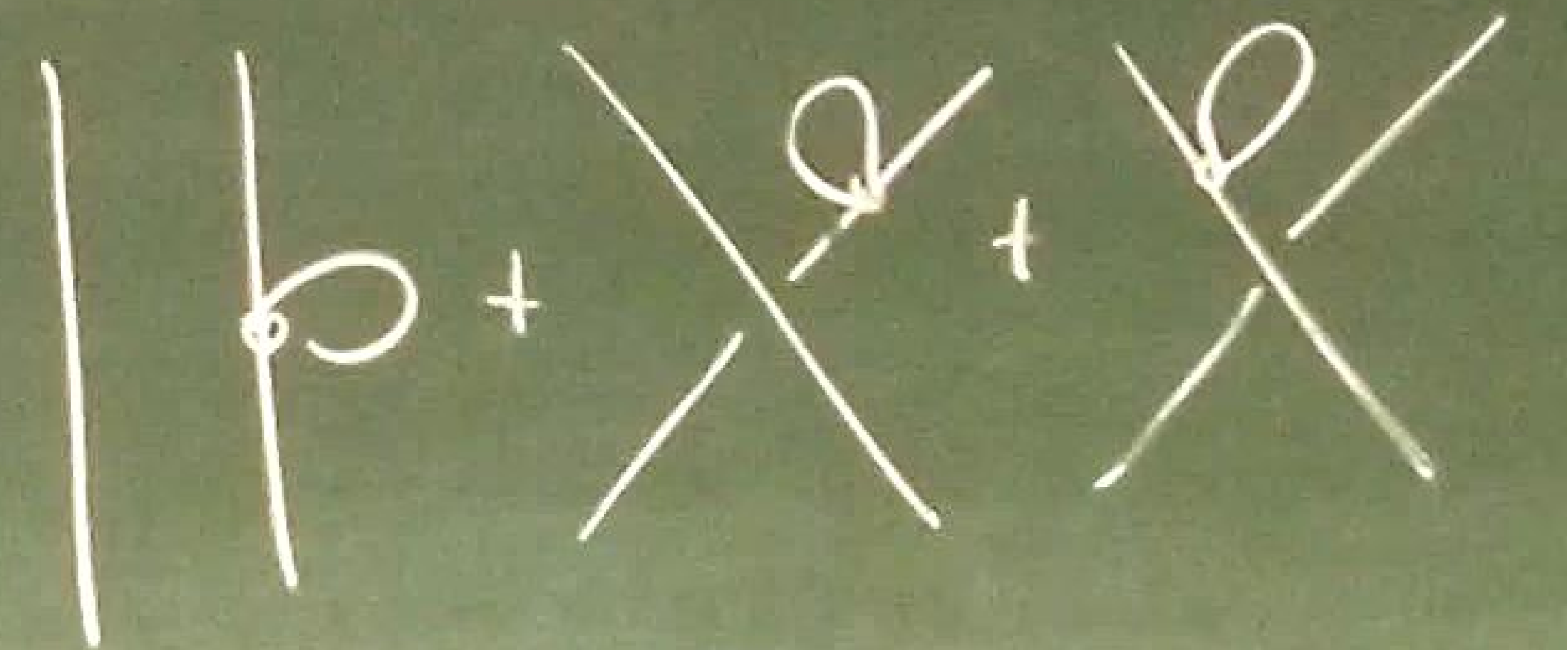
Example:

$$\langle \vec p_1 \vec p_2 | \vec p_A \vec p_B \rangle_c = \langle p_1 p_2 | p_A p_B \rangle + \langle p_1 p_2 | p_A p_B \rangle = (*)$$

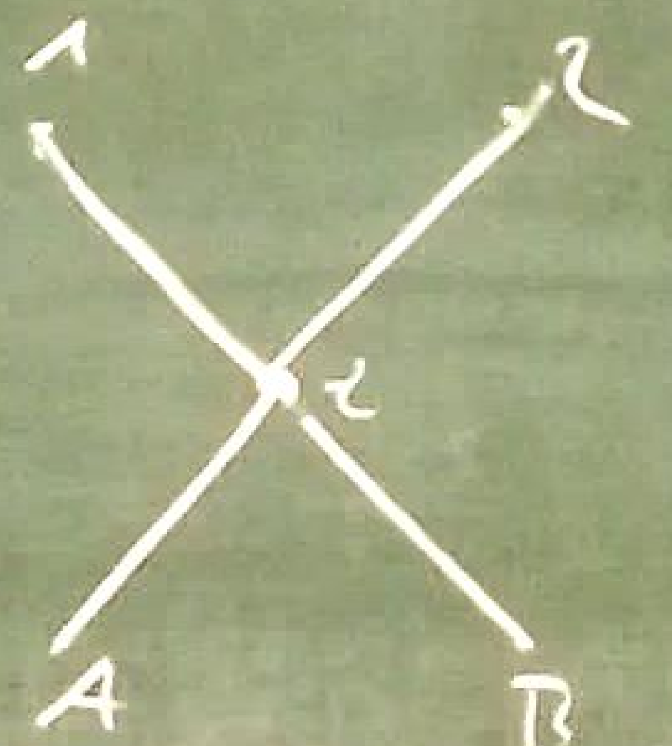
V) $-i\frac{\lambda}{4!}\int d^4x \; {}_0\langle p_1 p_2 | T\, \phi_I^4(x) | p_A p_B \rangle_0$

$=$ 

$\langle p p | \phi\phi\phi\phi | p p \rangle$

$+ \; \langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$ $+$ 

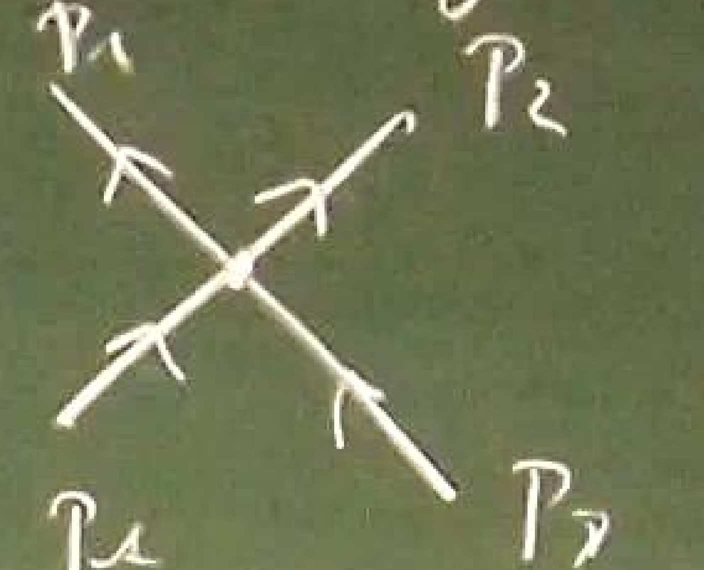
$+ \; \langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$



fully connected

→ Terms with $\phi\phi\phi\phi$ and $\phi\phi\phi\phi$ do not contribute to T

→ only fully connected diagrams contribute

VI) $\langle p_1 p_2 | iT | p_A p_B \rangle$ = (1st order) 

$$= \cancel{4!}\left(-i\frac{\lambda}{\cancel{4!}}\right)\int d^4x \; e^{-i(p_A+p_B-p_1-p_2)x}$$

$$= (-i\lambda)(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$$\overset{\text{def}}{=} i\,\boxed{M}\cdot(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$\rightarrow \mathcal{M}(p_A p_B \rightarrow p_1 p_2) = -\lambda + O(\lambda^2)$

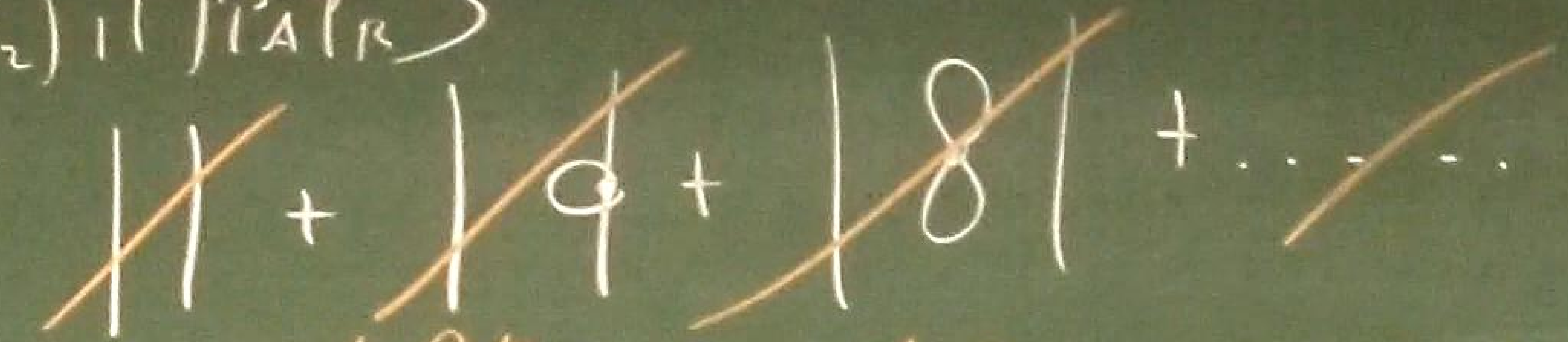
$$\rightarrow \sigma_{total} = \frac{\lambda^2}{32\pi E_{cm}^2}$$

$$\uparrow$$

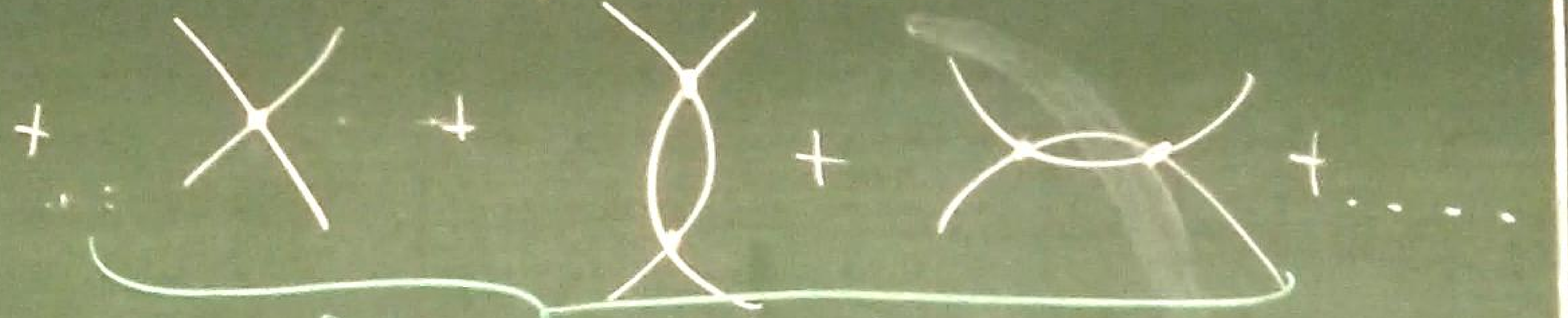
$$\int d\Omega \frac{d\sigma}{d\Omega} = 4\pi$$

3) Higher-order contributions

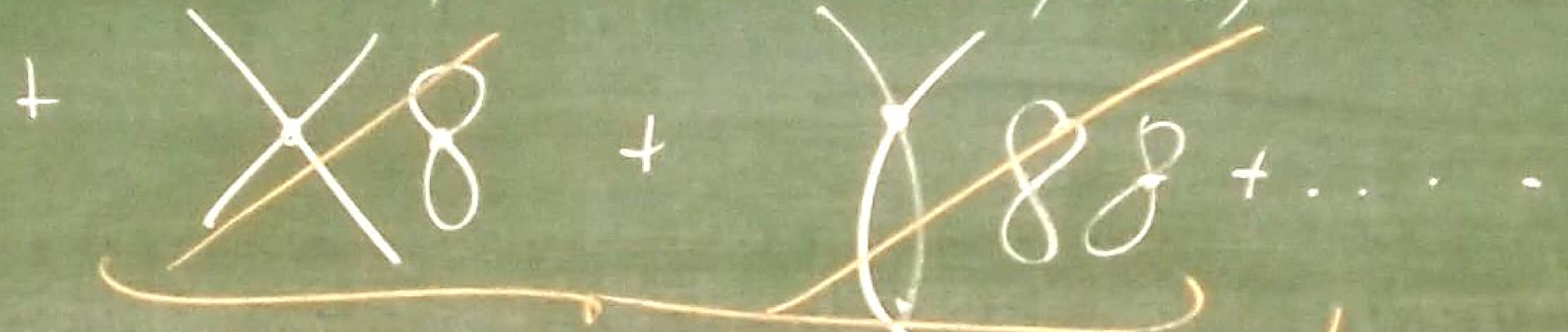
$\langle p_1 p_2 | iT | p_A p_B \rangle$

$=$ 

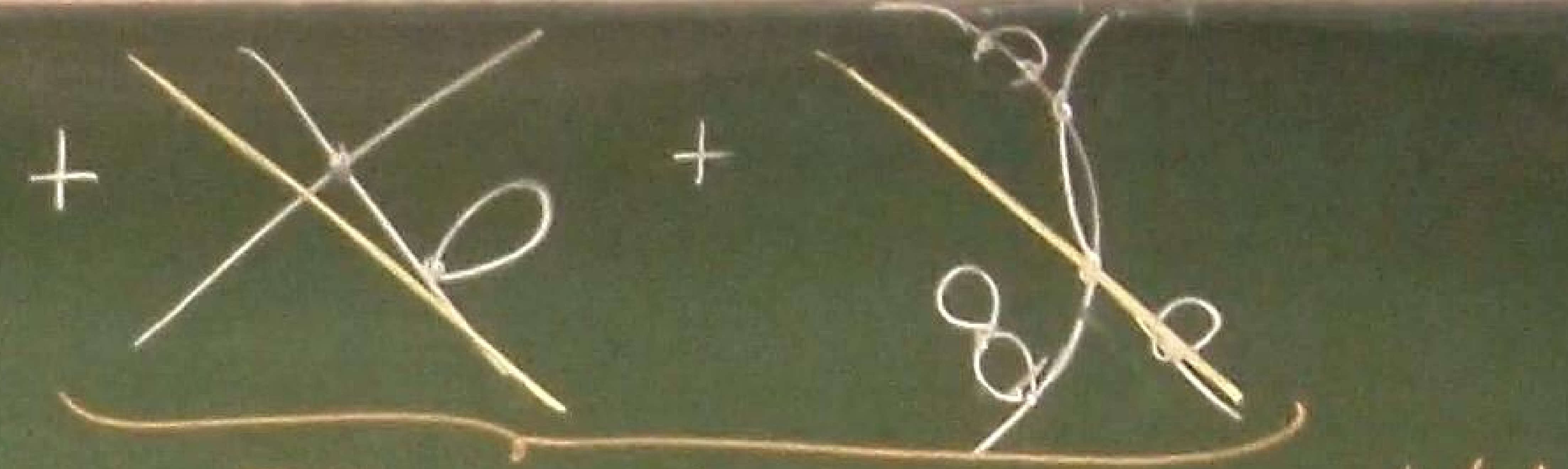
not fully connected



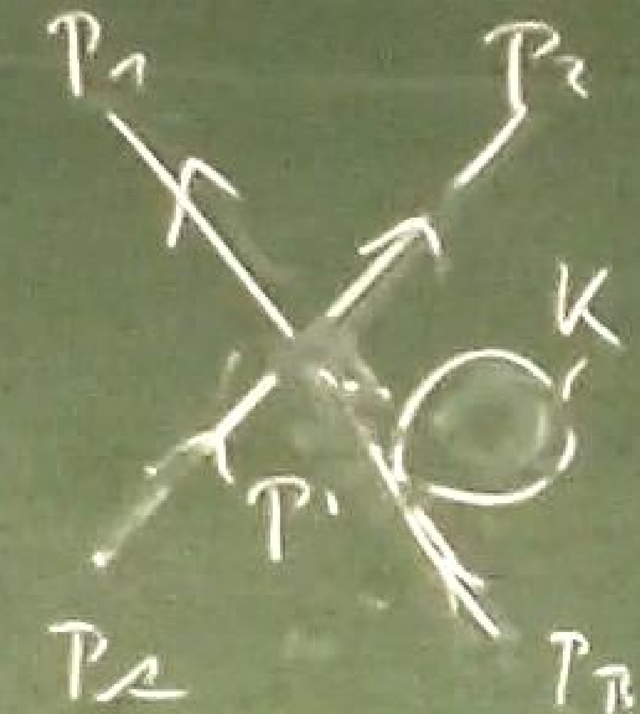
fully connected (amputated)



Bubbles exponentiate and drop out



Fully connected but not amputated



$$\doteq \frac{1}{2} \int \frac{d^4p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2}$$

$$\cdot (-i\lambda)(2\pi)^4 \delta(p_A + p' - p_1 - p_2)$$

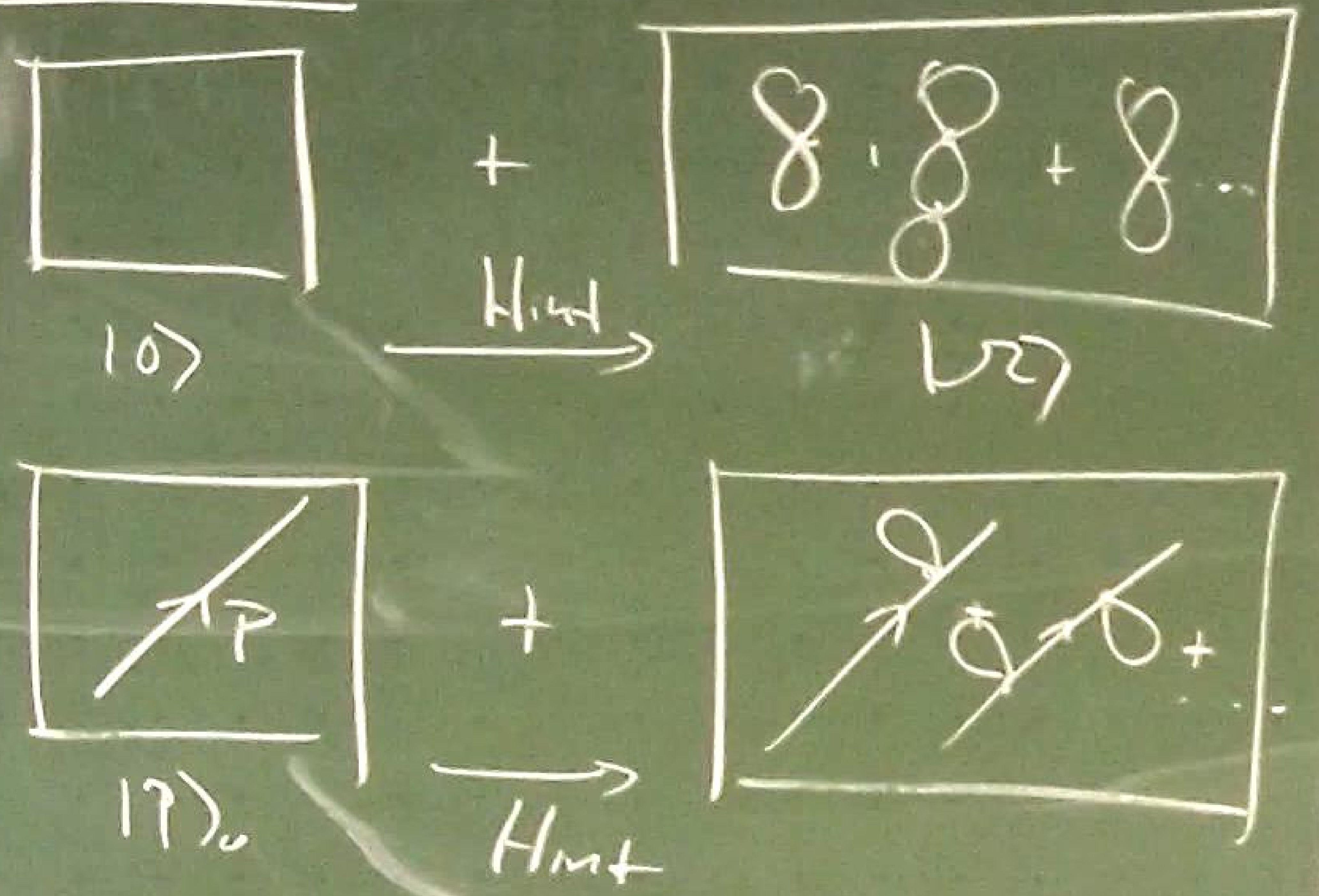
$$\cdot (-i\lambda)(2\pi)^4 \delta(p_B - p')$$

$$\sim \frac{-i}{p_B^2 - m^2 + i\epsilon} = \frac{1}{0} = \infty$$

$p_B^2 = m^2 \qquad E_p = p^0 = \sqrt{\vec{p}^2 + m^2}$

→ Eq (4.132) makes only sense without these diagrams!

! Interpretation:

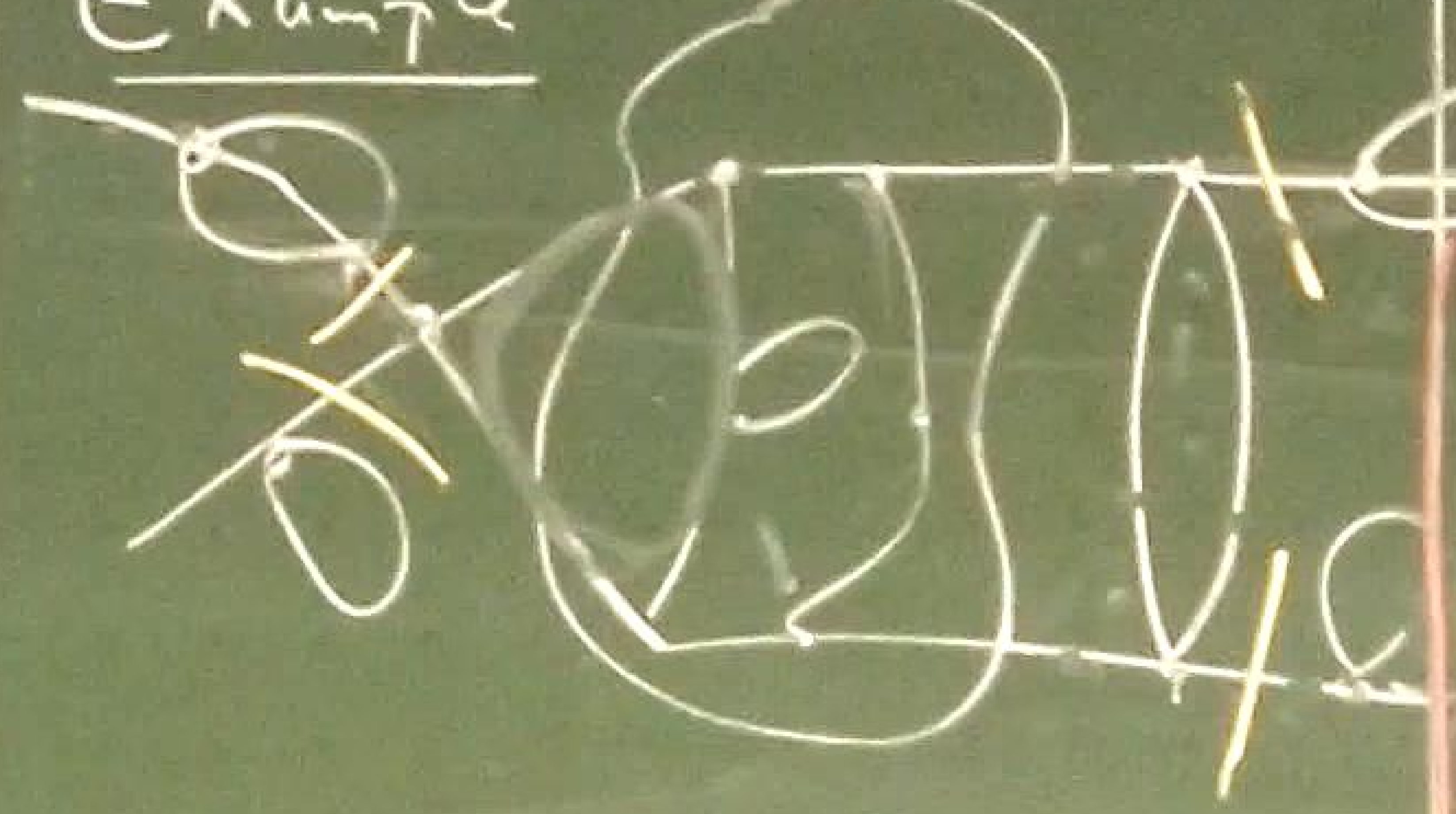


→ Not related to scattering!

→ Drop non-amputated diagrams

4] Amputation of diagrams

Example



Momentum space Feynman rules

1. Edges $\xrightarrow{p} = \frac{i}{p^2-m^2+i\epsilon}$
2. Vertex $= (-i\lambda)(2\pi)^4\,\delta(p_1+p_2-p_3-p_4)$
3. External lines $= 1$
4. Integrate over all internal momenta $\prod_i \int \frac{d^4p_i}{(2\pi)^4}$
5. Divide sym factor $\frac{1}{S}$

5] $\boxed{4.132} = iM\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-\sum p_f)$

$= \left\{ \right.$ Sum of all fully connected, amputated Feynman diagrams with p_A, p_B incoming and $\{p_f\}$ outgoing $\left. \right\}$

6] Position-space Feynman rules

1. For each edge x — y $= D_F(x-y)$
2. For each vertex $\times_z = (-i\lambda)\int d^4z$
3. For each external line $= e^{-ipz}$
4. Divide by sym factor: $\frac{1}{S}$