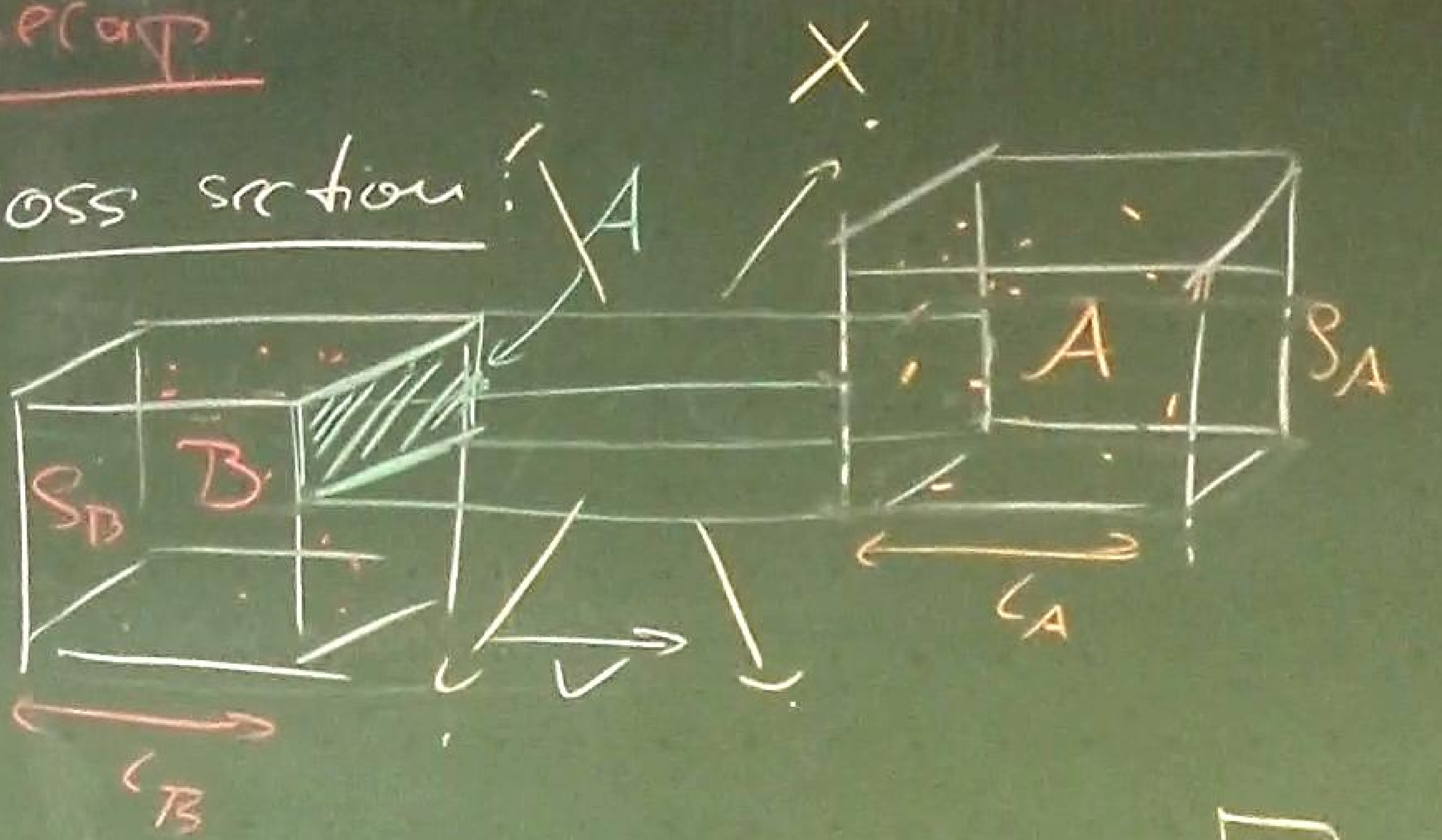
Recap:

Cross section:



$$\sigma_X = \frac{\#\text{ scattering events with outcome } \boxed{X}}{\underbrace{|A \rho_A l_A|}_{N_A}\,\underbrace{|\rho_B l_B A|}_{N_B}} \quad : A$$

The S-Matrix

1| * One-particle wavepacket

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k) |\vec{k}\rangle$$

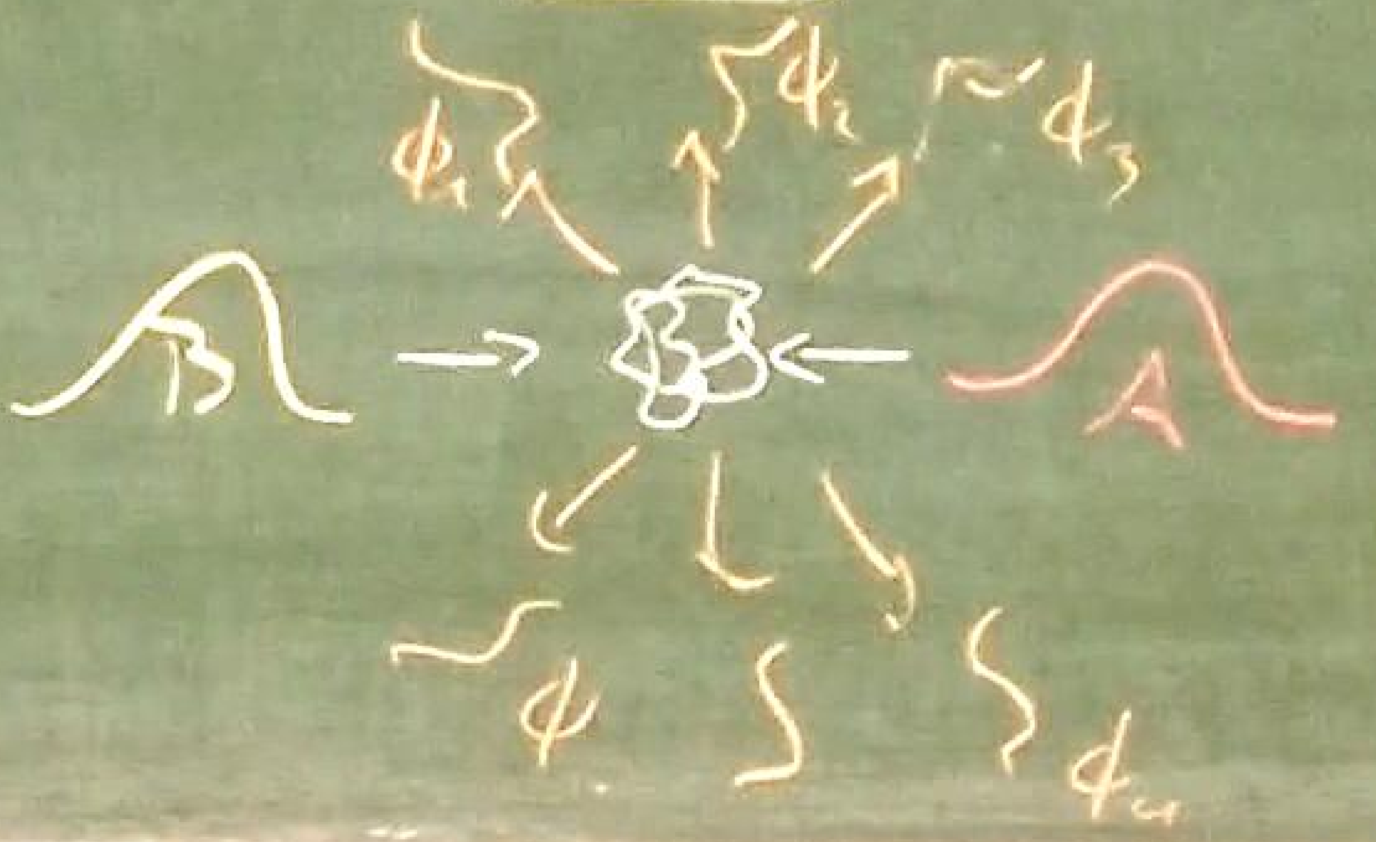
$$\langle\phi|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} |\phi(k)|^2 = 1$$

$|\vec{k}\rangle$: one-particle state of interacting theory

$\lambda=0 \to |\vec{k}\rangle_0 = \sqrt{2E_k}\, a_k^\dagger |0\rangle$

2| Probability

$$P = \left|{}_{out}\langle \phi_1 \dots \phi_n | \phi_A \phi_B \rangle_{in}\right|^2 =$$

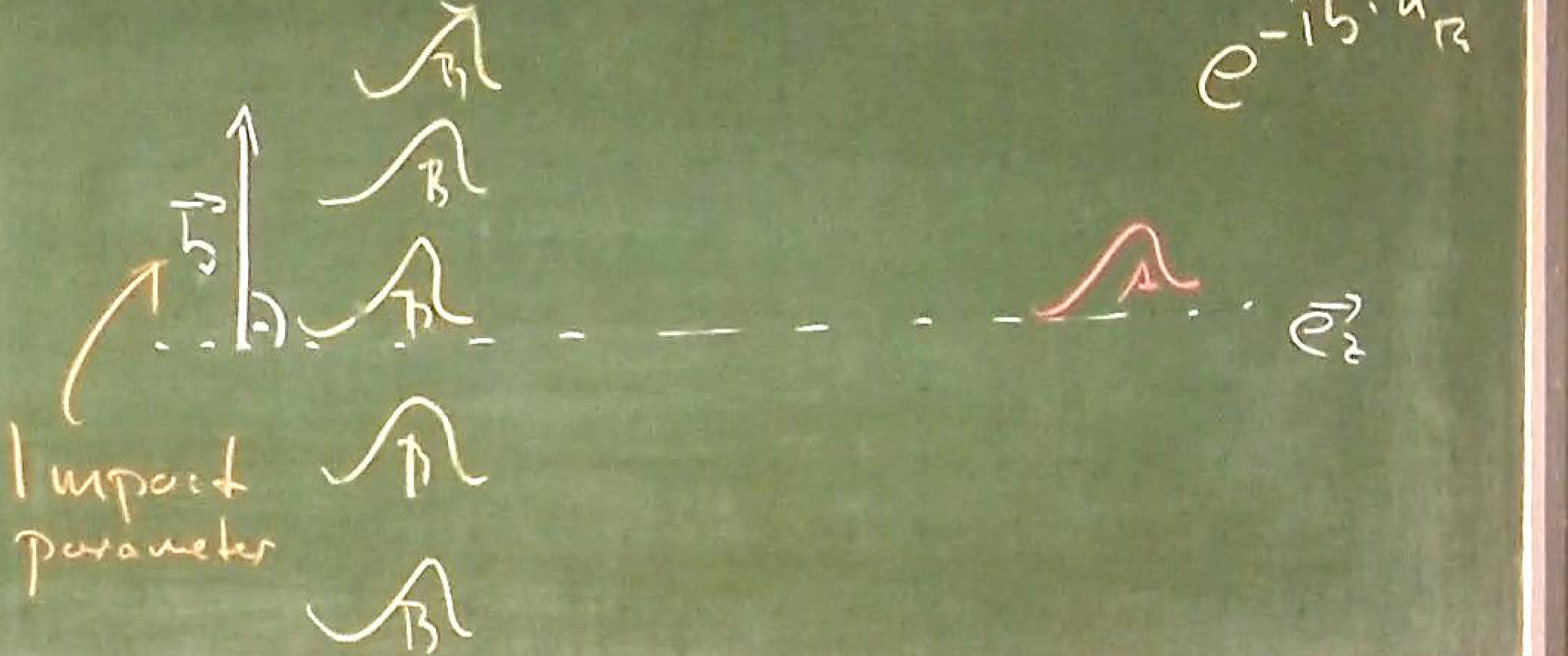


· $|\phi_A \phi_B\rangle_{in}$: in-state ($T \to -\infty$ two separate wavepackets)

· $|\phi_1 \dots \phi_n\rangle_{out}$: out-state ($T \to +\infty$ n separate wavepackets)

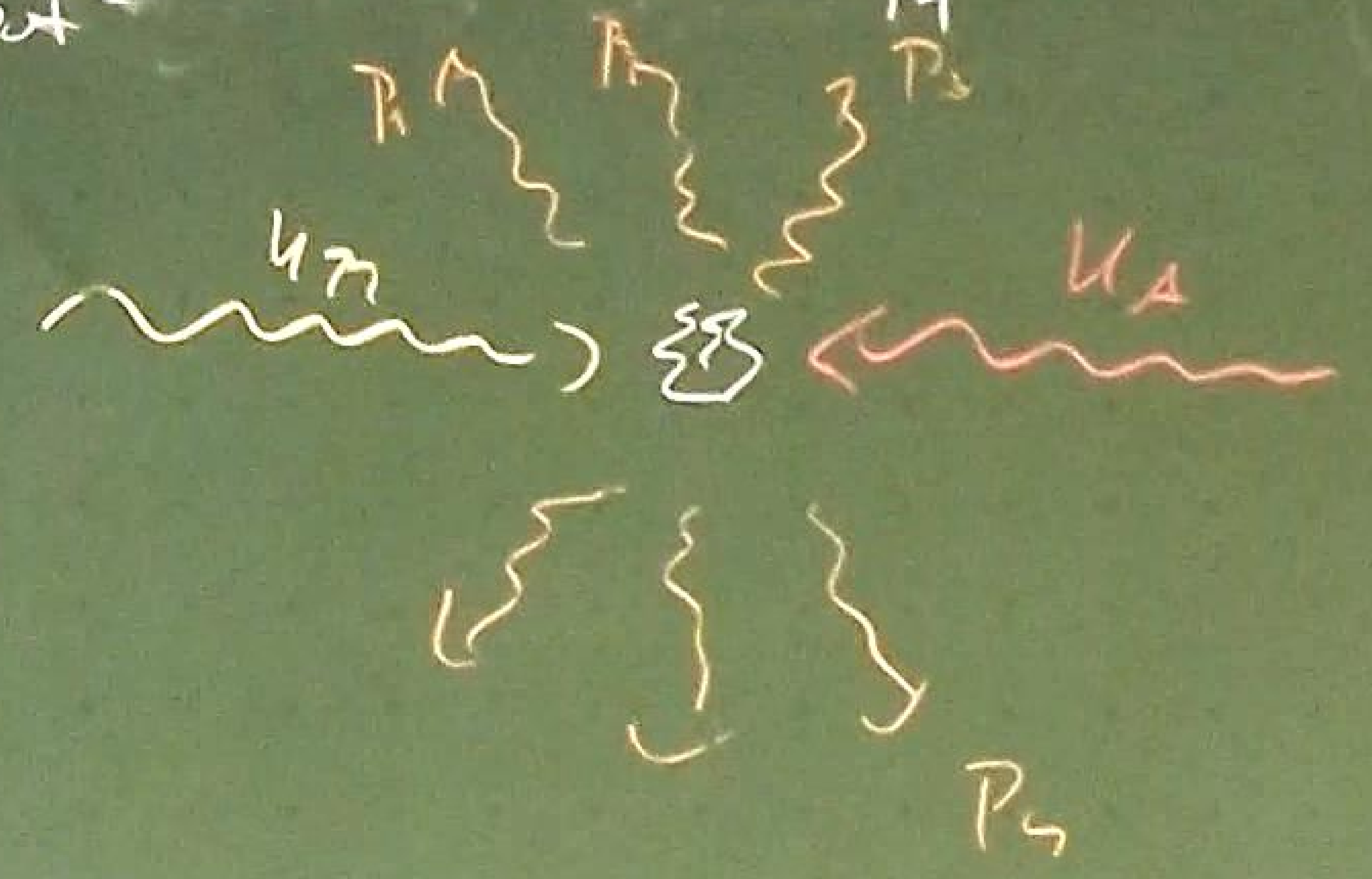
3| Fourier transformation

$$|\phi_A \phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(k_A)\phi_B(k_B)}{\sqrt{2E_{k_A}\, 2E_{k_B}}}\, e^{-i\vec{b}\cdot\vec{k}_B} |k_A k_B\rangle_{in}$$



4] $|\phi_1 \ldots \phi_n\rangle_{out} \to |\vec{p}_1 \ldots \vec{p}_n\rangle_{out}$

$\Rightarrow \; {}_{out}\langle p_1 \ldots p_n | k_A k_B \rangle_{in}$

$=$ 

5] S-matrix

$${}_{out}\langle \vec{p}_1 \ldots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{in} := \lim_{T\to\infty} {}_T\langle \vec{p}_1 \ldots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{-T}$$

$$= \lim_{T\to\infty} {}_{t_0=0}\langle p_1 \ldots p_n | e^{-i2TH} | k_A k_B \rangle_{t_0=0}$$

$$= {}_0\langle p_1 \ldots p_n | S | k_A k_B \rangle_0$$

Heisenberg state

$$P(t) |k\rangle_t = k\, |k\rangle_t$$

$$|k_t\rangle = e^{iHt} |k\rangle_0$$

$$|k_A k_B\rangle_{-T} = e^{-iHT} |k\rangle_{t_0=0}$$

$$|p_1 \ldots\rangle_T = e^{iHT} |p_1 \ldots p_n\rangle_{t_0=0}$$

Example: $S = \mathbb{1}$ for free theory

6] T-matrix:

$$\boxed{S = \underbrace{\mathbb{1}}_{\text{particles miss}} + \underbrace{iT}_{\text{non-trivial scattering}}}$$

7] $\to$ 4-momentum conservation

$p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$

$$\langle \vec{p}_1 \ldots \vec{p}_n | iT | \vec{k}_A \vec{k}_B \rangle$$

$$= (2\pi)^4 \delta^{(4)}\Big(k_A + k_B - \sum_f p_f\Big) \cdot i\, \underbrace{\mathcal{M}(k_A k_B \to \{p_f\})}_{\text{Invariant matrix element}}$$

Two questions:

1. $\mathcal{M} = ?$
2. $\boxed{\sigma = \sigma(\mathcal{M})}$

8] $\propto$ Probability to scatter $d\mathcal{V}_p = \prod_f d^3 p_f$

$$dP(AB_{\vec b} \to 1 \ldots n) = \underbrace{\left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{p_f}}\right)}_{\text{normalization}} \left| {}_{out}\langle \vec p_1 \ldots \vec p_n | \phi_A \phi_B(\vec b) \rangle_{in} \right|^2$$

9] $\propto$ Single target A and many incident particles $B_{\vec b}$

$$d(\#\text{ scattering events}) = \int d^2 b \, n_B \, dP(AB_{\vec b} \to 1 \ldots n)$$

n_B: Area density of B-particles

$$d\sigma = \frac{d(\# \ldots)}{\underbrace{n_B}_{\rho_B l_B} \cdot \underbrace{1}_{\rho_A l_A \cdot A}} = \int d^2 b \, dP(AB_{\vec b} \to 1 \ldots n) =$$

$$\Theta\left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{p_f}}\right) \int d^2 b \prod_{i=A,B} \left(\int \frac{d^3 k_i}{(2\pi)^3} \frac{\phi_i(\vec k_i)}{\sqrt{2E_{k_i}}} \int \frac{d^3 q_i}{(2\pi)^3} \frac{\phi_i^*(\vec q_i)}{\sqrt{2E_{q_i}}} \right)$$

$$\times \underbrace{e^{i\vec b(\vec q_B - \vec k_B)}}_{(2\pi)^2 \delta^{(2)}(k_B^\perp - q_B^\perp)} \; \underbrace{{}_{out}\langle \{p_f\} | \{k_i\} \rangle_{in}}_{iM(\{k_i\} \to \{p_f\}) \cdot (2\pi)^4 \delta^{(4)}(\sum k_i - \sum p_f)} \cdot \underbrace{{}_{out}\langle \{p_f\} | \{q_i\} \rangle_{in}^*}_{-iM^*(\{k_i\} \to \{p_f\}) \, (2\pi)^4 \delta^{(4)}(\sum q_i - \sum p_f)}$$

$$\boxed{q_i^\perp = (q_i^x, q_i^y)}$$

$\to$ Evaluate 6 q_i-integrals.

i) $q_B^\perp = k_B^\perp$

ii) $\to$ $q_A^\perp = k_A^\perp$

$$k_A^x + k_B^x = q_A^x + q_B^x$$

$$E_A^k + E_B^k = E_A^q + E_B^q$$

iii) ∢ $q_A^z\, q_B^z$-integrals:

$$\sqrt{\vec p_A^{\,2}+m^2}$$

$$\int dq_A^z\, dq_B^z\, \delta\Big(q_A^z+q_B^z-\sum p_f^z\Big)\,\delta\Big(E_A+E_B-\sum E_f\Big)$$

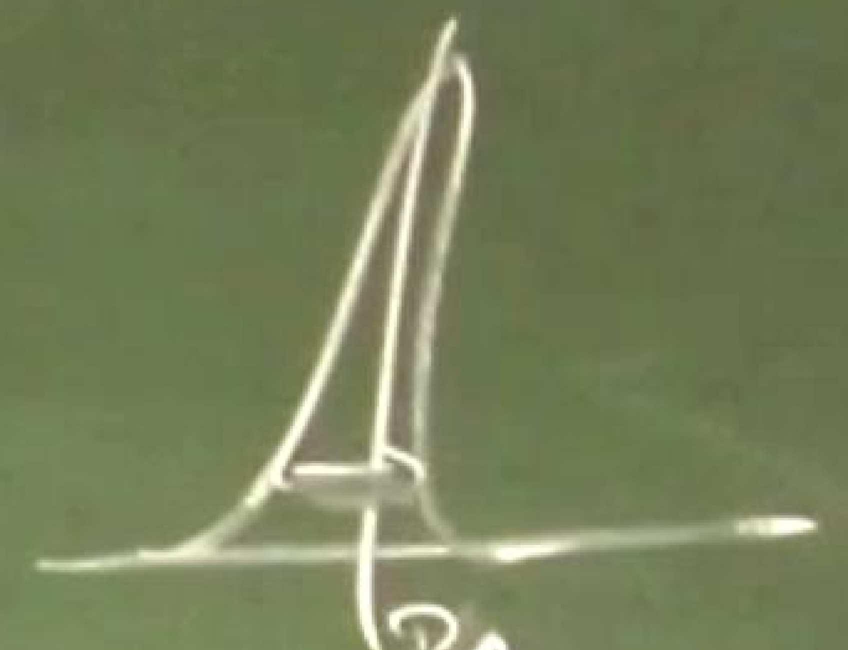
$$q_B^z=\sum p_f^z-q_A^z$$

$$=\int dq_A^z\,\delta\Big(\underbrace{\sqrt{q_A^{z\,2}+m_A^2}+\sqrt{q_B^{z\,2}+m_B^2}-\sum E_f}_{g(q_A^z)}\Big)\Bigg|_{q_B^z=\sum p_f^z-q_A^z}$$

$$=\frac{1}{|g'(q_A^z)|}\Bigg|_{g(q_A^z)=0}\ \hat{=}\ \frac{1}{\Big|\underbrace{\frac{q_A^z}{E_A}}_{v_A}-\underbrace{\frac{q_B^z}{E_B}}_{v_B}\Big|}=\frac{1}{|v_A-v_B|}$$

$$\boxed{v_{group}=\frac{\partial E(\vec q)}{\partial \vec q}=\frac{\vec q}{E_{\vec q}}}$$

10) $\phi_i(\vec k_i)$ peaked around $\vec p_A$, $\vec p_B$



$p_A^{(\mu)}$

$$d\sigma\approx\left(\prod_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{p_f}}\right)\frac{|\mathcal M(p_Ap_B\to\{p_f\})|^2}{2E_{p_A}2E_{p_B}|v_A-v_B|}$$

$$\underbrace{\int d^3k_A \ldots \int d^3k_B \ldots |\phi_A(k_A)|^2}_{1}\ \underbrace{|\phi_B(k_B)|^2}_{1}(2\pi)^4\delta^{(4)}\Big(\underbrace{k_A+k_B}_{p_A+p_B}-\sum p_f\Big)$$

11)

$$\boxed{d\sigma=\frac{1}{2E_{p_A}2E_{p_B}|v_A-v_B|}\left(\prod_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal M(p_Ap_B\to p_f)|^2\times(2\pi)^4\delta\Big(p_A+p_B-\sum p_f\Big)}$$

Special cases:

(2) Two final particles (p_1, p_2)

↪ center of mass frame $\vec{p}_A + \vec{p}_B = 0 = \vec{p}_1 + \vec{p}_2$

$$\boxed{\frac{d\sigma}{d\Omega} \overset{\circ}{=} \frac{1}{2E_{p_A} E_{p_B} |v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 \, 4 E_{cm}} |\mathcal{M}(p_A p_B \to p_1 p_2)|^2}$$

center of mass energy

(3) $m_A = m_B = m_1 = m_2$

$$\boxed{\frac{d\sigma}{d\Omega} \overset{\circ}{=} \frac{|\mathcal{M}(p_A p_B \to p_1 p_2)|^2}{64 \pi^2 E_{cm}^2}}$$

$$E_{cm} = [E_{p_A} + E_{p_B}]_{cm} = \sqrt{(p_A + p_B)^2}$$