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Problem 9.1: Principal axes of inertia

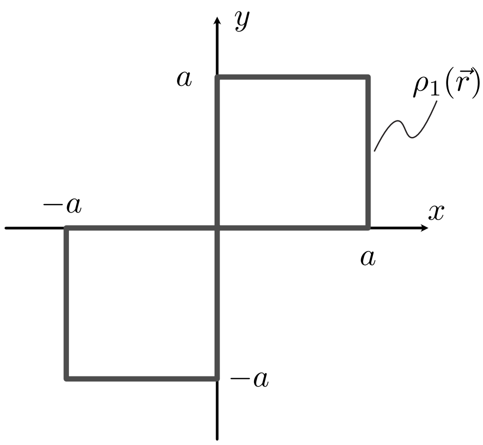
[**Written** | 3 pt(s)]

ID: ex_principal_axes_of_inertia:km25

Learning objective

For a planar mass distribution, we determine the inertia tensor explicitly and thus determine the principal axes of inertia and the principal moments of inertia.

We consider a homogeneous thin wire of mass $8m$, which is shaped in the x-y plane as shown in the sketch.



a) What is the mass density $\rho(\boldsymbol{r})$ for the mass distribution? Calculate the inertia tensor I with respect to the center of mass in the given coordinate system. Why does $I_{zz} = I_{xx} + I_{yy}$ apply to a planar mass distribution? 1pt(s)

Hint: The mass density for section 1 of the wire (see sketch) is,

$$\rho_1(\boldsymbol{r}) = \frac{m}{a}\delta(z)\delta(x-a)\Theta(y)\Theta(a-y),$$

with the Heaviside function $\Theta(x)$. The result for the inertia tensor is

$$I = ma^2 \begin{pmatrix} \frac{10}{3} & -2 & 0 \\ -2 & \frac{10}{3} & 0 \\ 0 & 0 & \frac{20}{3} \end{pmatrix}.$$

b) Calculate the principal moments of inertia. 1pt(s)

c) Determine the principal axes of inertia. 1pt(s)

Problem 9.2: Billiards [**Oral** | 3 pt(s)]

ID: ex_billiards:km25

Learning objective

With our knowledge of rigid bodies, we would now like to understand the physics behind a billiard shot in more detail.

We consider a billiard ball of mass M and radius R with a homogeneous mass distribution.

a) Determine the moment of inertia of the billiard ball. **1**pt(s)

The billiard ball is now struck with a cue (centered) so that the center of gravity of the ball has the speed v_0 and the ball does not rotate at the beginning. The strength of the friction between the table and the ball is given by $Mg\mu$ with the coefficient of friction μ.

b) How far does the ball move before the initial sliding motion changes to a pure rolling motion? **1**pt(s)

c) We now want to prevent the ball from sliding at the beginning. Where must the billiard ball be hit with the cue so that it directly performs a pure rolling movement? **1**pt(s)

Problem 9.3: Precession of a force-free spinning top [**Oral** | 3 pt(s)]

ID: ex_force_free_top:km25

Learning objective

In the lecture, the precessional motion of a symmetrical spinning top in earth's gravitational field was derived. Here we want to show that a force-free spinning top also performs a precessional motion if the axis of rotation does not coincide with one of the stable principal axes of inertia.

The change in angular momentum $\boldsymbol{M}$ in the body-fixed system is described by the Euler equations,

$$\frac{d}{dt}\boldsymbol{M} = \boldsymbol{N} + \boldsymbol{M} \times \boldsymbol{\Omega}, \tag{1}$$

where $\boldsymbol{\Omega}$ is the angular velocity in the body-fixed system and $\boldsymbol{N}$ is the external torque. In the following, we consider a force-free ($\boldsymbol{N} = 0$) symmetrical spinning top with the principal moments of inertia $I_1 = I_2 \neq I_3$.

a) Solve the Euler equations in the principal axis system. Sketch the movement of $\boldsymbol{\Omega}$. **1**pt(s)
Hint: The angular momentum satisfies $\boldsymbol{M} = I\boldsymbol{\Omega}$.

In the moving coordinate system, we can express $\boldsymbol{\Omega}$ by the Euler angles,

$$\Omega_1 = \dot{\vartheta}\cos\psi + \dot{\varphi}\sin\psi\sin\vartheta \tag{2}$$

$$\Omega_2 = -\dot{\vartheta}\sin\psi + \dot{\varphi}\cos\psi\sin\vartheta \tag{3}$$

$$\Omega_3 = \dot{\psi} + \dot{\varphi}\cos\vartheta. \tag{4}$$

b) Use the solution from a) to determine the time dependence of the Euler angles $\vartheta(t)$, $\psi(t)$ and $\varphi(t)$. **1**pt(s)

Hint: Use the fact that the angular momentum $\boldsymbol{m}$ is conserved in the laboratory system and choose $\boldsymbol{m} \parallel \boldsymbol{e}_z$. Express $\boldsymbol{e}_z$ in terms of the unit vectors in the body-fixed system.

c) Make a sketch of the movement of the spinning top. **1**pt(s)