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Problem 5.1: Harmonic central potential in 3D

[**Oral** | 3 pt(s)]

ID: ex_harmonic_oscillator_3D:km25

Learning objective

In the lecture we saw that the $1/r$ potential can be solved analytically. We now repeat the calculations for the harmonic potential r^2.

Consider a particle in three dimensions in a harmonic central potential $V = \frac{1}{2}m\omega^2\boldsymbol{r}^2$. Apply the methods from the lecture to the general solution of the central potential.

a) Use conservation of angular momentum to map the problem to a plane and introduce polar coordinates. **1**pt(s)

b) Derive the formal solution for $r(t)$ and $r(\varphi)$. **1**pt(s)

c) Both integrals can be solved analytically. Calculate the integrals and determine $r(t)$ and the trajectory $r(\varphi)$ and $\varphi(t)$. **1**pt(s)

Problem 5.2: Laplace-Runge-Lenz vector

[**Written** | 2 (+3 bonus) pt(s)]

ID: ex_laplace_runge_lenz_en:km25

Learning objective

In this task we consider the Laplace-Runge-Lenz vector, another conserved variable of the $1/r$ potential in addition to energy and angular momentum.

We first consider a particle of mass m in the gravitational potential $V = -k/r$. The corresponding equation of motion is

$$m\ddot{\boldsymbol{r}} = -\frac{k}{r^2}\hat{\boldsymbol{r}} \quad \text{with} \quad k = GMm. \tag{1}$$

Where $\hat{\boldsymbol{r}}$ is the unit vector pointing in the direction $\boldsymbol{r}$, i.e. $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$

a) Show that the Laplace-Runge-Lenz vector $\boldsymbol{A}$, defined as **1**pt(s)

$$\boldsymbol{A} = m\dot{\boldsymbol{r}} \times \boldsymbol{L} - mk\hat{\boldsymbol{r}}, \tag{2}$$

is a constant of motion. What is the geometric interpretation of this conserved variable?

b) How is the magnitude of the Laplace-Runge-Lenz vector related to the eccentricity ϵ of Kepler's orbit? **1**pt(s)

Hint: First, calculate $\boldsymbol{A}\boldsymbol{r}$.

From Noether's theorem, we know that every symmetry leads a conserved variable. In the same way, every conserved variable, including the Laplace-Runge-Lenz vector, corresponds to a symmetry of the Lagrangian $L = T - V$. We will show this in the following. First, consider the infinitesimal transformation

$$q_i' = q_i + \delta q_i \quad \text{with} \quad \delta q_i = m \sum_{j=1}^{3} \epsilon_j \left(2\dot{q}_i q_j - q_i \dot{q}_j - \boldsymbol{q}\dot{\boldsymbol{q}}\delta_{ij}\right), \tag{3}$$

with the parameters ϵ_j. The time is not transformed, so the velocities transform as follows

$$\dot{q}_i' = \dot{q}_i + \frac{d}{dt}\delta q_i. \tag{4}$$

*c) Calculate to the first order of ϵ_j the change of Lagrangian under this transformation. The result has the form **+2pt(s)**

$$L\left(q_i', \dot{q}_i'\right) = L\left(q_i, \dot{q}_i\right) + g(q_i, \dot{q}_i, \ddot{q}_i) + \mathcal{O}(\epsilon_i^2). \tag{5}$$

Then consider the function

$$f_j = m\left(m\dot{\boldsymbol{q}}^2 q_j - m\boldsymbol{q}\dot{\boldsymbol{q}}\dot{q}_j + k\frac{q_j}{q}\right). \tag{6}$$

Calculate $\frac{d}{dt}f_j$, and conclude that the Lagrangian only changes by one total derivative under the infinitesimal transformation.

*d) Now use Noether's theorem to show that the Laplace-Runge-Lenz vector is the conserved quantity associated with this symmetry. **+1pt(s)**

Problem 5.3: Perihelion rotation

[**Oral** | 3 pt(s)]

ID: ex_perihelion_precession:km25

Learning objective

In this task we consider the perihelion rotation. Classically, this is caused by a perturbation potential; in the theory of relativity, it is obtained naturally by additional corrections to Newton's limiting case.

In the gravitational potential of the sun $V_0 = -k/r$ the planets move in elliptical orbits. The perihelion describes the closest point of the elliptical orbit to the sun and does not change in the unperturbed gravitational potential. However, if an additional perturbation potential $V = V_0 + \delta V(r)$ is considered, this leads to a perihelion rotation.

In the unperturbed gravitational potential, the angle between one perihelion and the next perihelion changes by exactly 2π. With the perturbation, the change in angle is calculated via

$$\Delta\varphi = -2\sqrt{2\mu}\frac{d}{dl}\int_{r_{min}}^{r_{max}} dr \sqrt{E - V(r) - \frac{l^2}{2\mu r^2}}, \tag{7}$$

where μ is the reduced mass and l is the absolute value of angular momentum.

a) First check whether for $\delta V = 0$ equation (7) gives $\Delta\varphi = 2\pi$. **1**pt(s)

Hint: Take a derivative and use substitutions from the lecture notes

b) Calculate the perihelion rotation $\delta\varphi = \Delta\varphi - 2\pi$ in the first order of δV. The result is **1**pt(s)

$$\delta\varphi = 2\mu \frac{d}{dl} \left[\frac{1}{l} \int_0^\pi d\varphi \, r^2(\varphi) \, \delta V(r(\varphi)) \right]. \tag{8}$$

Where $r(\varphi)$ is the unperturbed solution (i.e. an ellipse).

c) Find the result for the perturbation potentials $\delta V = \gamma/r^3$ and $\delta V = \beta/r^2$. **1**pt(s)