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Problem 4.1: Noether's theorem

[**Written** | 3 pt(s)]

ID: ex_KM_noether_theorem_en2:km25

Learning objective

In this exercise we apply Noether's theorem to a Galilean boost. For this example, the transformed Lagrange functions differ by a total derivative, which gives an additional term in the invariant quantity.

a) The Lagrange function of a free particle is $L(\boldsymbol{r}, \dot{\boldsymbol{r}}) = m\dot{\boldsymbol{r}}^2/2$. Carry out a Galilean boost $\boldsymbol{r}' = \boldsymbol{r} + s\boldsymbol{v}t$ and determine the new Lagrange function. Show that it differs from the original Lagrange function by only one total derivative dF/dt. **1**pt(s)

b) If a transformation changes the Lagrange functions by only one total derivative **1**pt(s)

$$L(q^i, \dot{q}^i, t) = L'(h^i, \dot{h}^i, t) = L(h^i, \dot{h}^i, t) + \frac{d}{dt}F(h^i, t),$$

with h^i the transformed coordinates, then the Euler-Lagrange equations of L and L' coincide. The Galilean boost from a) is such a symmetry of the equations of motion. Noether's theorem now gives the conserved quantity

$$I(q^i, \dot{q}^i, t) = \left[\sum_j \left(\frac{\partial L(q^i, \dot{q}^i, t)}{\partial \dot{q}^j}\frac{\partial q^j(h^i, t, s)}{\partial s}\right) - \frac{\partial F(h^i, t, s)}{\partial s}\right]_{s=0}.$$

Determine this for the free particle from a) and show that the conserved quantity results in a rectilinear motion. Does the invariance of the equations of motion under Galilean boosts always result in rectilinear motion in general?

c) Prove Noether's theorem for transformations that change the Lagrange function by a total derivative. **1**pt(s)

Hints: Make sure that still

$$\left.\frac{\partial L'}{\partial s}\right|_{s=0} = \frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}^i}\frac{\partial q^i}{\partial s}\right]_{s=0}$$

applies, as in the proof from the lecture. Then determine dI/dt.

Problem 4.2: Conserved quantities [**Oral** | 3 pt(s)]

ID: ex_KM_conserved_quantities:km25

> **Learning objective**
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> Energy, momentum and angular momentum are the most important and most frequent conserved quantities that occur in physical systems. In this exercise, you will practice how to read off these conserved quantities from the potentials.

For the following cases, determine whether energy, momentum or angular momentum are conserved. Also state for which spatial direction conservation of momentum applies and about which axes of rotation conservation of angular momentum applies. Here $g, D, w > 0$ are given constants.

a) A particle is located in an external force field that is described by the potential $\phi(\boldsymbol{r}, t)$. **1** pt(s)

   I) $\phi(\boldsymbol{r}, t) = mgz$

   II) $\phi(\boldsymbol{r}, t) = \frac{1}{2} D \cos(\omega t) \boldsymbol{r}^2$

b) Two particles 1 and 2 of the same mass exert a force on each other with the interaction potential $\phi(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$. There is no external force field. Momentum and angular momentum here mean the total momentum and total angular momentum respectively: $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$, $\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2$. **1** pt(s)

   I) $\phi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \frac{1}{2} D \exp(-\omega t)(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2$

   II) $\phi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \frac{1}{2} D(\boldsymbol{r}_1 + \boldsymbol{r}_2)^2$

   III) $\phi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \frac{1}{2} D((x_1 - x_2)^2 + 2(y_1 - y_2)^2)$

c) Search for further conserved quantities for the given potentials. **1** pt(s)

   Hint: Remember also the previous task.

Problem 4.3: Electromagnetic field [**Oral** | 4 pt(s)]

ID: ex_KM_electromagnetic_field:km25

> **Learning objective**
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> In this exercise we will practice the Lagrange formalism for charged particles in electromagnetic fields. In particular, we will see that in these systems the kinetic and canonical momentum typically no longer coincide and only one of them can be conserved.

A particle with mass m and charge q moves in the electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$ under the Lorentz force

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}.$$

The fields are given by the vector potential $\boldsymbol{A}$ and the scalar potential ϕ:

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}, \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

a) Consider a particle in a homogonous magnetic field along the z-axis $\boldsymbol{B} = B\boldsymbol{e}_z$. Show that $\boldsymbol{B}$ is given by the potentials $\phi = 0$, $\boldsymbol{A} = -yB\boldsymbol{e}_x$. Show further that the particle trajectory **1pt(s)**

$$\boldsymbol{r}(t) = \begin{pmatrix} \cos(\omega_c t) \\ -\sin(\omega_c t) \\ 0 \end{pmatrix},$$

with the cyclotron frequency $\omega_c = qB/mc$, is a solution of the equations of motion.

b) The Lagrange function for a particle in a magnetic and electric field takes the form **1pt(s)**

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \frac{m}{2}\dot{\boldsymbol{r}}^2 + \frac{q}{c}\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}, t) - q\phi(\boldsymbol{r}, t)$$

Demonstrate by careful calculations that the Euler-Lagrange equation gives rise to the Lorentz force.

c) Determine the canonical momentum. How does this differ from the kinetic momentum $m\dot{\boldsymbol{r}}$? **1pt(s)**

d) For a charged particle in the homogeneous magnetic field from a), the Lagrange function is translation invariant in the x-direction, i.e. $\boldsymbol{r} \mapsto \boldsymbol{r} + s\boldsymbol{e}_x$ is a symmetry. Calculate the conserved quantity using Noether's theorem and check that this quantity is actually conserved for the special trajectory from a). **1pt(s)**