Prof. Dr. Hans-Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

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Problem 3.1: Harmonic oscillator in two dimensions

[**Oral** | 4 pt(s)]

ID: ex_harmonic_oscillator_in_two_dimensions:km25

> **Learning objective**
>
> This task offers a simple application of the Lagrange formalism using the example of the two-dimensional harmonic oscillator. Different coordinate systems are used.

A mass m is located in a 2-dimensional isotropic harmonic potential

$$V(\boldsymbol{x}) = \frac{D}{2}\boldsymbol{x}^2. \tag{1}$$

a) Determine the kinetic energy T, the potential energy V and the Lagrange function $\mathcal{L} = T - V$ of the particle in Cartesian coordinates. **1**pt(s)

b) Determine the Euler-Lagrange equations in Cartesian coordinates and solve them for the general boundary conditions $\boldsymbol{x}(t=0) = \boldsymbol{x}_0$, $\dot{\boldsymbol{x}}(t=0) = \boldsymbol{v}_0$. **1**pt(s)

c) Now switch to polar coordinates, set up the Lagrange function again and show that there is a cyclic variable. Which variable is a conservation variable? **1**pt(s)

d) Set up the Euler-Lagrange equations in polar coordinates. **1**pt(s)

Problem 3.2: Calculus of variations: Geodesics

[**Written** | 4 pt(s)]

ID: ex_variational_calculation_geodesics:km25

> **Learning objective**
>
> In this task we use a simple calculus of variations to find the shortest connection between two points on curved surfaces.

We consider a space curve $\boldsymbol{\gamma}$ in three dimensions, which is parameterized by $t \in [t_1, t_2]$. The arc length of this space curve is defined as

$$L = \int_{\gamma} \mathrm{d}s = \int_{t_1}^{t_2} |\dot{\gamma}(t)| \,\mathrm{d}t = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t. \tag{2}$$

a) Show that the arc length is invariant under transformations of the parameterization $t \to t'$. **1**pt(s)

A geodesic describes the shortest connection between two points.

b) Use a calculus of variations to show that the shortest connection between two points in three-dimensional space is a straight line. **1**pt(s)

Hint: Since the arc length is independent of the parametrization, a favourable choice can be made here.

c) Now consider a cylindrical shell **1**pt(s)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\,\cos\varphi \\ R\,\sin\varphi \\ z \end{pmatrix} \tag{3}$$

with $(\varphi, z) \in [0, 2\pi] \times [0, l]$. Parameterize a curve on the cylindrical surface by $z = z(t)$, $\varphi = \varphi(t)$ over a parameter range $t \in [t_1, t_2]$. Use a calculus of variations to find the shortest connection between two points on a cylindrical surface.

d) Now consider a spherical surface **1**pt(s)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix} \tag{4}$$

with $(\varphi, \vartheta) \in [0, 2\pi] \times [0, \pi]$. Parameterize a curve on the surface of the sphere by $\varphi = \varphi(t)$, $\vartheta = \vartheta(t)$ over a parameter range $t \in [t_1, t_2]$. To find the shortest connection between two points on the surface of the sphere, first derive the Euler-Lagrange equations. Find the solution for the special case of curves whose end points are located on the spherical equator at $\vartheta(t_1) = \vartheta(t_2) = \pi/2$.

Hint: Show that movements in the equatorial plane solve the Euler-Lagrange equations.

Problem 3.3: Waste disposal on board the ISS

[**Oral** | 3 pt(s)]

ID: ex_waste_disposal_iss:km25

Learning objective

In this task, an equation of motion with a small perturbation is solved using a development.

The ISS approximately describes a circular orbit around the earth, with radius R and angular frequency ω. At the time $t = 0$, the astronauts throw a waste bag with relative initial velocity v_0 in the direction of the earth. In the approximation $v_0 \ll \omega R$ it can be assumed that: $r(t) = R + r_1(t)$ with $r_1(t) \ll R$ and $\phi(t) = \omega t + \phi_1(t)$ with $\phi_1(t) \ll 2\pi$.

Determine and solve the equations of motion for r_1 and ϕ_1 up to the first order of the perturbation.