Prof. Dr. Hans-Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

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Problem 2.1: Conservative force fields

[**Written** | 4 pt(s)]

ID: ex_conservative_force_fields:km25

> **Learning objective**
>
> Using three examples, we examine whether the given force fields are conservative and repeat some mathematical basics, that will be needed for the rest of the lecture.

We start with the two force fields

$$\boldsymbol{K}_1 = x^2\boldsymbol{e}_x + xy\boldsymbol{e}_y + \boldsymbol{e}_z\,, \tag{1}$$
$$\boldsymbol{K}_2 = (y-4x)\boldsymbol{e}_x + (x-2y)\boldsymbol{e}_y. \tag{2}$$

a) Investigate whether the force fields are conservative. Determine the scalar potential $V_i(x,y,z)$ for all conservative force fields $\boldsymbol{K}_i$. **1**pt(s)

b) What work is done by $\boldsymbol{K}_1$ or $\boldsymbol{K}_2$ during a revolution around the unit square $[0,1]\times[0,1]$? **1**pt(s)

In the following we look at the force field

$$\boldsymbol{K}_3 = \frac{-y}{x^2+y^2}\,\boldsymbol{e}_x + \frac{x}{x^2+y^2}\,\boldsymbol{e}_y\,. \tag{3}$$

c) Make a sketch of the force field. **1**pt(s)

d) Determine the rotation of $\boldsymbol{K}_3$. Does a potential $V(x,y)$ exist with $\boldsymbol{K}_3 = -\nabla V(x,y)$? Explicitly calculate the work integral for $\boldsymbol{K}_3$ along the unit circle around the origin. Is the force field $\boldsymbol{K}_3$ conservative? **1**pt(s)

Problem 2.2: Motion in one dimension

[**Oral** | 3 pt(s)]

ID: ex_motion_in_one_dimension:km25

> **Learning objective**
>
> In this task, we examine a one-dimensional potential and see how we can obtain information about the behavior of the system using approximate methods.

Investigate the one-dimensional motion of a particle with the coordinate x (in dimensionless units) in the potential

$$V(x) = 3x^2 - 2x^3. \tag{4}$$

a) Formulate the equation of motion and find the stationary solutions. Show that the stationary solutions coincide with the extrema of the potential. **1**pt(s)

b) Expand the potential around each of the two extrema into a 2nd order Taylor series in x. Solve the approximate equations of motion. For which point does a stable solution exist? **1**pt(s)

c) Sketch the phase space diagram for the system. Draw the motion trajectories in the (x,p)-plane for different energies. The results from b) can be very helpful here. **1**pt(s)

Problem 2.3: Motion in one dimension 2

[**Oral** | 4 pt(s)]

ID: ex_motion_in_one_dimension_2:km25

Learning objective

In this exercise, we derive useful relations for one-dimensional systems and further familiarize ourselves with the properties of phase space portraits.

The energy of a one-dimensional particle in the time-independent potential $V(x)$ (in dimensionless units) is given as

$$E = \frac{1}{2}\dot{x}^2 + V(x). \tag{5}$$

a) Show by calculating the total derivative dE/dt, that the energy is conserved. **1**pt(s)

b) Show that the equation of motion can be written as **1**pt(s)

$$\dot{x} = \pm\sqrt{2(E - V(x))}. \tag{6}$$

Find the formal solution to this differential equation.

c) Show that the trajectories in the (x,p)-plane are just the curves that belong to a fixed value of the energy. Sketch the solutions in the phase space diagram for the following potentials **1**pt(s)

$$V(x) = x^4 - 2x^2, \quad V(x) = \cos(x), \tag{7}$$

for the energies $E = -1,\ E = -0.5,\ E = 0,\ E = 0.5,\ E = 1$ and $E = 2$.

d) For a periodic solution, let the "action" $S(E)$ be defined as the area enclosed by the curve with energy E in the (x,p) plane. Show that the period of the motion T can be written as **1**pt(s)

$$T = \frac{d}{dE}S(E). \tag{8}$$