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Problem 13.1: Rod in the garage

[**Oral** | 4 pt(s)]

ID: rod_in_garage:km25

Learning objective

In this exercise, we will examine a common paradox in more detail. If you consider a rod with high speed and a garage with the same length, you can close both sides of the garage at the same time while the rod flies through it, because the rod is shortened due to the length contraction from the reference system of the garage. Conversely, however, the garage is also shortened from the reference system of the rod and the rod cannot fit into the garage with the doors closed.

We consider a rod of rest length L that moves towards a garage at speed $\boldsymbol{v} = v\boldsymbol{e}_x$. The garage also has the length L and a door at the entrance and exit. We close door 1 as soon as the end of the rod has passed the start of the garage and open door 2 as soon as the start of the rod reaches the end of the garage.

a) In the following, we first consider the inertial system (IS) K in which the garage rests. Specify the start and end points of the garage and the rod in IS K as functions $x(t)$. At what time is door 2 opened? 1pt(s)

   Hint: Choose the IS K so that at time $t = 0$ the end of the rod is at door 1 at position $x = 0$.

b) Now we switch to IS K' in which the rod rests. Write the start and end points of the garage and the rod in K' as functions $x'(t')$. 1pt(s)

   Hint: Choose also the IS K' so that at time $t' = 0$ the end of the rod is at door 1 at position $x' = 0$.

To solve the paradox and better understand the thought experiment, we consider four space-time points (t, x):

(i) The end of the rod reaches door 1, and this closes: $(0, 0)$

(ii) The start of the rod at time $t = 0$: $(0, L/\gamma)$

(iii) The door 2 at time $t = 0$: $(0, L)$.

(iv) The start of the rod reaches door 2 and this opens: $(\frac{L}{v}(1 - \frac{1}{\gamma}), L)$

c) Transform the four space-time points into the IS K' to (t', x'). 1pt(s)

d) Now draw a space-time diagram for both IS K and K' (t over x or t' over x') with the world lines of the rod and the garage and mark the four space-time points. 1pt(s)

   Interpret the result.

Problem 13.2: Relativistic collision

[**Oral** | 4 pt(s)]

ID: relativistic_collision:km25

> **Learning objective**
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> In this task, we will familiarize ourselves even more with relativistic mechanics using the example of an elastic collision.

We consider the elastic collision (i.e. the masses before and after the collision are conserved) of two masses m_1 and m_2. In the inertial frame K, mass 1 is at rest and mass 2 is moving with the speed $\boldsymbol{v}$ (with $v = |\boldsymbol{v}|$) towards mass 1. After the collision, mass 1 has the velocity $\boldsymbol{v}_1$ and mass 2 has the velocity $\boldsymbol{v}_2$.

a) What is the relativistic conservation of energy and momentum? Write the corresponding four-vectors explicitly. **1**pt(s)

The collision process is not yet fully defined by the initial velocities, but also depends on an collision parameter. We now want to consider the collision process independently of the collision parameter and instead calculate it depending on the magnitude of the velocity $v_2 = |\boldsymbol{v}_2|$ after the collision.

b) Specify the magnitude of the velocity of mass 1 after the collision $v_1 = |\boldsymbol{v}_1|$ as a function of the initial velocity v and the velocity v_2. **1**pt(s)

   Hint: Use the relativistic conservation of energy.

The amount of velocity v_1 is now also known.

c) Now we are interested in the angle θ between the velocities of the two masses after the collision. Specify this as a function of the two velocities after the collision (v_1 and v_2). **1**pt(s)

   Hint: Use the fact that the scalar product $p_1^\mu p_{2\,\mu}$ is conserved before and after the collision (why?). Also use the relation $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = v_1 v_2 \cos\theta$.

   intermediate result:

$$\cos\theta = \frac{c^2}{v_1 v_2} \frac{(\gamma_1 - 1)(\gamma_2 - \frac{m_1}{m_2})}{\gamma_1 \gamma_2} \tag{1}$$

d) Finally, we want to look at special limiting cases of the result from (c). What is the angle θ in the following cases **1**pt(s)

   (i) In the non-relativistic limiting case of small velocities (**Hint:** $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}(\frac{v^4}{c^4})$).

   (ii) In the case of equal masses $m_1 = m_2$. Consider this case first in the non-relativistic limit case, how does the angle change at relativistic velocities?

   (iii) In the limiting case $v_1 \to c$ and $v_2 \to c$.

Problem 13.3: Four-speed and four-acceleration [**Oral** | 2 pt(s)]

ID: four_speed_four_acceleration:km25

> **Learning objective**
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> In this task, we will familiarize ourselves with the expansion of velocity and acceleration in the form of a four-vector.

We start with the four-vector

$$x^\mu = \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} . \tag{2}$$

The four-velocity u^μ and the four-acceleration b^μ are then defined as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \quad \text{and} \quad b^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} , \tag{3}$$

where τ is the proper time with $\mathrm{d}\tau = \mathrm{d}t/\gamma$. The advantage of this definition of velocity and acceleration is that they can be transformed from one inertial frame to another via the Lorentz transformations, analogous to the four-vector.

a) Calculate the four-vector velocity as a function of the velocity vector $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$. What is the scalar product of u^μ with itself? **1** pt(s)

b) Calculate the four-acceleration as a function of the velocity vector $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ and the acceleration $\dot{\boldsymbol{v}} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$. Show that in Minkowski space the four-acceleration is always orthogonal to the four-velocity. **1** pt(s)