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Problem 12.1: Lorentz transformation [**Written** | 4 pt(s)]

ID: ex_KM_Lorentz_transformation:km25

> **Learning objective**
>
> In this task, we will take a closer look at the Lorentz transformation using the example of an arbitrary Lorentz boost. By applying this Lorentz transformation, we derive the relativistic addition of two velocities.

a) For an arbitrary velocity vector $\boldsymbol{v}$, with $v = |\boldsymbol{v}|$, show that the Lorentz transformation is given by **1**pt(s)

$$
\begin{aligned}
ct' &= \gamma ct - \gamma \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{c} \\
\boldsymbol{x}' &= \boldsymbol{x} + (\gamma - 1) \frac{\boldsymbol{x} \cdot \boldsymbol{v}}{v^2} \boldsymbol{v} - \gamma \boldsymbol{v} t \,.
\end{aligned} \tag{1}
$$

Hint: Begin by applying the transformations to the components of $\boldsymbol{x}$ that are parallel ($\boldsymbol{x}_{\parallel}$) and perpendicular ($\boldsymbol{x}_{\perp}$) to the velocity vector $\boldsymbol{v}$, and then derive the transformation for the full vector $\boldsymbol{x}$.

b) Use the result (1) to determine the corresponding 4×4 matrix representation Λ of this Lorentz transformation so that **1**pt(s)

$$
\begin{pmatrix} ct' \\ \boldsymbol{x}' \end{pmatrix} = \lambda \cdot \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} \,. \tag{2}
$$

c) A particle now moves in the inertial frame (IS) K' with the velocity $\boldsymbol{u}' = \mathrm{d}\boldsymbol{x}'/\mathrm{d}t'$. What velocity $\boldsymbol{u} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ has the particle in IS K? **1**pt(s)

Hint: use $\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}/\mathrm{d}t'}{\mathrm{d}t/\mathrm{d}t'}$.

result:

$$
\boldsymbol{u} = \frac{1}{\gamma} \frac{\boldsymbol{u}' + (\gamma - 1)\frac{\boldsymbol{u}' \cdot \boldsymbol{v}}{v^2}\boldsymbol{v} + \gamma \boldsymbol{v}}{1 + \frac{\boldsymbol{u}' \cdot \boldsymbol{v}}{c^2}} \tag{3}
$$

d) Now consider the two special cases $\boldsymbol{u}' \perp \boldsymbol{v}$ and $\boldsymbol{u}' \parallel \boldsymbol{v}$, and calculate $\boldsymbol{u}$ for these cases. In both cases, make sure that for speeds $|\boldsymbol{u}'| < c$ and $|\boldsymbol{v}| < c$ the speed $|\boldsymbol{u}|$ is always less than the speed of light c. **1**pt(s)

Problem 12.2: Generating function for harmonic oscillator [**Oral** | 3 pt(s)]

ID: ex_KM_generating_func_harmonic_oscillator:km25

> **Learning objective**
>
> Appropriate canonical transformations can drastically simplify the form of a given Hamiltonian function. As a concrete example, let's look at the harmonic oscillator.

a) Specify the Hamiltonian function of the harmonic oscillator (mass m, natural frequency ω, generalized coordinate q). **1pt(s)**

b) Perform a canonical transformation with the generating function **1pt(s)**

$$F_2(q,P) = \frac{m}{2}\omega q\sqrt{\frac{2P}{m\omega} - q^2} + P\,\arcsin\left(q\sqrt{\frac{m\omega}{2P}}\right). \tag{4}$$

What are the canonical equations of the oscillator in the new variables? Solve them and find $Q(t)$ and $P(t)$. Also determine $q(t)$ and $p(t)$ by back-transforming $Q(t)$ and $P(t)$. Sketch the trajectories in (q,p) and (Q,P) phase space.

c) Determine the generating function $F_1(q,Q)$ corresponding to $F_2(q,P)$. **1pt(s)**

Problem 12.3: Time dilation and twin paradox [**Oral** | 3 pt(s)]

ID: ex_KM_time_dilation_and_twin_paradox:km25

> **Learning objective**
>
> In this task, we look at time dilation using the example of muons. Based on this, we solve another paradox of the special theory of relativity: the twin paradox.

We consider two sets of muons (A and B), both of which are located at point 1 at time $t = 0$. in the inertial frame (IS) K, the muons of set A are at rest. The muons of set B move with a velocity $v = 0.9995c$ from point 1 to point 2 (distance $d = 100$m). The mean lifetime of the muons (in the muon's rest system) is $\tau = 2.2 \cdot 10^{-6}$ s.

In IS K, the number of muons in set A is given as $A(t) = A_0 \mathrm{e}^{-t/\tau}$ and, analogously, in IS K' (rest system of set B), the number of muons in set B is given as $B(t') = B_0 \mathrm{e}^{-t'/\tau}$. At time $t = t' = 0$ there are $A_0 = B_0 = 1000$ muons in both sets.

a) Describe the number of muons in set B from the point of view of IS K ($B(t)$). How many muons are there in sets A and B at time t_{P2}, when array B reaches point 2? **1pt(s)**

What is the effective lifetime τ_{eff} of the moving muons B in the stationary system K? Compare the distance traveled by a muon during the effective lifetime with the distance traveled if muons would behave non-relativistically.

In the previous part of the task, we learned why muons can travel much longer distances than naively expected despite their short lifespan.

Now we want to look at the twin paradox. To do this, we place a magnet at point 2, which reverses the direction of flight of the muons in set B so that set B flies back to point 1. We assume that the time required to reverse the direction of flight can be neglected.

b) Now consider, in the IS K, the number of muons in set A and B during the complete flight time. How many muons will be in sets A and B when set B arrives back at point 1? **1**pt(s)

c) Finally, let's look at the rest frame of set B. Explain why this is no longer an inertial frame. **1**pt(s)

We now define the outbound path as the time up to directly before the reversal of direction and the return path as the time from directly after the reversal of direction. From the perspective of B, what percentage of the muons in sets A and B decay on the outbound and return paths?

Are the calculated values identical to part (b)? If not, explain where the discrepancy comes from.