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Juni 19th, 2025
SS 2025

Problem 11.1: Poisson bracket

[**Written** | 3 pt(s)]

ID: ex_KM_Poisson_bracket:km25

> **Learning objective**
>
> In this exercise, we prove the properties of the Poisson bracket and familiarize ourselves with it.

a) Show the following properties of the Poisson bracket: **1**pt(s)

- Product rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \tag{1}$$

- Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \tag{2}$$

b) Calculate $\{f, q_i\}$ and $\{f, p_i\}$. What follows from this for the fundamental Poisson brackets $\{q_i, q_j\}$, $\{p_i, p_j\}$ and $\{q_i, p_j\}$? **1**pt(s)

c) Consider a three-dimensional harmonic oscillator with mass m and natural frequency $\omega_x = \omega_y = \omega_z = \omega$. Set up the Hamiltonian function H and use the Poisson bracket to show that the second-order tensor **1**pt(s)

$$T_{ij} = p_i p_j + (m\omega)^2 q_i q_j \tag{3}$$

is a conserved quantity.

Problem 11.2: Hamilton function and Hamilton's equations of motion

[**Oral** | 3 pt(s)]

ID: ex_hamiltonian_function_and_equations_of_motion:km25

> **Learning objective**
>
> In this task we describe the symmetric gyroscope and a charged particle in the electromagnetic field using the Hamiltonian formalism.

a) In the lecture, the Lagrangian function of a symmetric gyroscope **1**pt(s)

$$L(\varphi, \vartheta, \psi, \dot\varphi, \dot\vartheta, \dot\psi) = \frac{I_1}{2}\left(\dot\vartheta^2 + \dot\varphi^2 \sin^2(\vartheta)\right) + \frac{I_3}{2}\left(\dot\psi + \dot\varphi\cos(\vartheta)\right)^2 - mgl\cos(\vartheta) \tag{4}$$

with the Euler angles φ, ϑ and ψ. Calculate the corresponding Hamilton function and Hamilton's equations of motion.

The Lagrange function of a charged particle in the electromagnetic field is

$$L = \frac{m}{2}\dot{\boldsymbol{r}}^2 - q\left(\phi(\boldsymbol{r},t) - \dot{\boldsymbol{r}}\cdot\boldsymbol{A}(\boldsymbol{r},t)\right). \tag{5}$$

b) Set up the Hamiltonian function H and determine the Hamiltonian equations of motion. Derive the equation of motion for the charged particle from this. **1**pt(s)

c) Investigate how H changes under a gauge transformation **1**pt(s)

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\xi \tag{6a}$$
$$\phi' = \phi - \partial\xi/\partial t. \tag{6b}$$

Determine the transformed Hamiltonian function and the corresponding equations of motion. What influence does the gauge transformation have on the motion of the charged particle?

Problem 11.3: Poisson brackets of the angular momentum [**Oral** | 1 pt(s)]

ID: ex_KM_Poisson_brackets_of_angular_momentum:km25

Learning objective

We look at angular momentum and see that its Poisson brackets obey exactly the angular momentum algebra. We will encounter this algebra again in quantum mechanics.

We consider a particle of mass m in a central potential $V(\boldsymbol{x})$ in three dimensions. The Lagrangian is given by $L = T - V$.

Determine the Hamiltonian function. Express the angular momentum $\boldsymbol{L}$ via the canonical variables $\boldsymbol{x}$ and $\boldsymbol{p}$ and show

$$\{L_i, L_j\} = \epsilon_{ijk} L_k,$$

the “commutation relations” of the angular momentum.