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Juni 17th, 2025
SS 2025

Problem 10.1: Unstable rotation

[**Oral** | 3 pt(s)]

ID: ex_unstable_rotation:km25

> **Learning objective**
>
> In this task we get a better intuition about unstable rotation axes of rigid bodies. We support our observations with a simulation.

A rigid body has the (pairwise different) principal moments of inertia I_1, I_2, I_3 and rotates freely with the angular momentum $\boldsymbol{M}$ (in the solid system), so that its kinetic energy E remains constant.

a) Show that the conservation of energy means that the angular momentum vector $\boldsymbol{M}$ is located on the surface of an ellipsoid. **1**pt(s)

Permitted angular momentum $\boldsymbol{M}(t)$ is defined by Euler's equations (without external torque)

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega} \tag{1}$$

and must be located at the intersection of the energy ellipsoid and the spherical surface, which is described by $|\boldsymbol{M}| = \text{const.}$. Here, $\boldsymbol{\Omega}$ is the angular velocity in the body-fixed system.

b) Plot the energy ellipsoid and the angular momentum sphere for the following initial conditions: **1**pt(s)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \boldsymbol{M}_1(t=0) = \begin{pmatrix} 1 \\ 0.1 \\ 0.1 \end{pmatrix}, \quad \boldsymbol{M}_2(t=0) = \begin{pmatrix} 0.1 \\ 1 \\ 0.1 \end{pmatrix}, \quad \boldsymbol{M}_3(t=0) = \begin{pmatrix} 0.1 \\ 0.1 \\ 1 \end{pmatrix} \tag{2}$$

Which initial angular momentum is followed by an unstable rotation?

c) Solve the Euler equations numerically for the initial conditions mentioned above and plot the solution in the figures generated in subtask b). **1**pt(s)

Hint: Proceed as in Problem 7.2 and use the Euler method with sufficiently small time steps to solve equation (1).

Problem 10.2: Legendre transformation

[**Written** | 3 pt(s)]

ID: ex_legendre_transform:km25

> **Learning objective**
>
> In this task, we will familiarize ourselves with the properties of the Legendre transformation.

The Legendre transform of the function f is given by

$$g(y) = xy - f(x) \tag{3}$$

with $y = f'(x)$.

a) Show that $f'' > 0$ or $f'' < 0$ is sufficient for the well-definedness of the transformation. 1pt(s)

b) Derive that the relationship $g'' > 0$ also follows from $f'' > 0$ and therefore the reverse transformation exists. 1pt(s)

c) Apply the Legendre transformation a second time and show that the original function $f(x)$ results again. 1pt(s)

Problem 10.3: Spinning Neutron Star

[**Oral** | 2 pt(s)]

ID: ex_spinning_top_neutron_star:km25

Learning objective

Neutron stars are known to sometimes have very violent starquakes, during which their mass distribution changes. This makes them a natural example for a spinning top with changing moments of inertia. In this exercise we will investigate how this change in inertia affects the rotation.

Assume that the surface of a Neutron star is vibrating slowly, such that the principal moments of inerta can be described by

$$I_{zz} = \frac{2}{5} m R^2 \left(1 + \epsilon \cos(\omega t)\right) , \tag{4}$$

$$I_{xx} = I_{yy} = \frac{2}{5} m R^2 \left(1 - \frac{\epsilon}{2} \cos(\omega t)\right) . \tag{5}$$

The deviation from the perfect sphere is small ($\epsilon \ll 1$). At the same time, the Neutron star is spinning with angular velocity $\mathbf{\Omega}(t)$.

a) Show that the z component of $\mathbf{\Omega}(t)$ stays almost constant. 1pt(s)

b) Determine the trajectory of $\mathbf{\Omega}(t)$ and sketch it. Consider especially the limiting cases of $\omega \to 0$ and $\epsilon \ll \omega / \Omega_z$. 1pt(s)