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December 11th, 2025
WS 2025/26

Problem 9.1: Time-Dependent Perturbation Theory

[**Written** | 2 (+1 bonus) pt(s)]

ID: ex_time_dependent_perturbation_theory:fqt2526

> **Learning objective**
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> The first subtask is an application of time-dependent perturbation theory in first order. The goal is to identify the correct transition amplitude and evaluate the integrations analytically. The second subtask is a challenging calculation to derive the *exact* solution of the problem—a useful repetition/application of important methods in quantum mechanics. This problem is one of the few examples where the time evolution can be solved analytically (which allows for a verification of perturbation theory).

We investigate a one-dimensional harmonic oscillator with mass m, charge e, and frequency ω in a time-dependent electric field $E(t)$. The Hamiltonian is of the form

$$
\begin{aligned}
& H = H_0 + H'(t)\,, \\
\text{where} \quad & H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \quad \text{(harmonic oscillator)} \\
\text{and} \quad & H'(t) = ex\,E(t) \quad \text{(perturbation)}\,.
\end{aligned} \tag{1}
$$

The time-dependency of the external electric field is given by

$$E(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2}\, \cos(\Omega t)\,, \tag{2}$$

where $A \in \mathbb{R}$ is a constant, $\tau > 0$ is a decay rate and $\Omega > 0$ is a frequency.

a) Calculate the transition probability $P_{0\to n}(t, t_0)$ from the ground state $|0\rangle$ at $t_0 \to -\infty$ to an excited state $|n\rangle$ at $t \to +\infty$ in first order perturbation theory. What happens for $\tau \to 0$? **1**pt(s)

Hint: Use $x = \sqrt{\frac{\hbar}{2m\omega}}\,\left(a^\dagger + a\right)$ to evaluate the matrix element.

b) The transition probability can also be calculated *exactly* using the following identity **1**pt(s)

$$\hat{T}\, e^{-i\int_{t_0}^{t} \mathrm{d}t'\,\left(f(t')a + f^*(t')a^\dagger\right)} = e^{-i\int_{t_0}^{t} \mathrm{d}t' f(t')a}\, e^{-i\int_{t_0}^{t} \mathrm{d}t' f^*(t')a^\dagger}\, e^{\int_{t_0}^{t} \mathrm{d}t' f^*(t') \int_{t_0}^{t'} \mathrm{d}t'' f(t'')} \tag{3}$$

which is a generalization of the well-known relation for the displacement operator. Determine the time evolution for the initial state $|0\rangle$, and show that the transition probabilities $P_{0\to n}(t, t_0)$ for $t_0 \to -\infty$ and $t \to +\infty$ take the form

$$P_{0\to n} = \frac{K^{2n}}{n!} e^{-K^2} \quad \text{with} \quad K = \frac{eA}{2\sqrt{2m\omega\hbar}} e^{-\frac{\tau^2(\omega+\Omega)^2}{4}} \left(1 + e^{\tau^2\omega\Omega}\right) \tag{4}$$

and compare the result with a).

*c) Prove Eq. (3). **+1**pt(s)

Hint: Apply the same method as the proof of the relation $e^{A+B} = e^A\, e^B\, e^{-[A,B]/2}$ requires.

Problem 9.2: Spontaneous decay of the Hydrogen Atom [**Oral** | 3 pt(s)]

ID: ex_spontaneous_decay_hydrogen_atom:fqt2526

Learning objective

In this exercise you will calculate which transitions in the Hydrogen atom can occur spontaneously. Based on this you then will calculate the transition rates for electrons from the $n = 2$ manifold back to the ground state.

Consider an excited Hydrogen atom in the state $|n, l, m\rangle$ inside the electromagnetic vacuum. We try to find the possible transitions into a state $|n', l', m'\rangle$ by emitting a photon in the mode $|n_{k,\lambda} = 1\rangle$. Assuming the light-matter interaction is adiabatically turned on and off we can calculate the transition in first order perturbation theory as

$$\langle n', l', m'; n_{k,\lambda} = 1 | \Psi(t)\rangle = \frac{1}{i\hbar} e^{-i\mathcal{E}_f (t-t_0)} \int_{t_0}^{t} dt_1 \,\langle n', l', m'; n_{k,\lambda} = 1 |\, H_{\text{int}}(t_1)\, | n, l, m; n_{k,\lambda} = 0\rangle \,. \tag{5}$$

In the lecture It was shown that this results in the transition rate

$$\Gamma = \frac{d}{dt} |\,\langle n', l', m'; n_{k,\lambda} = 1 | \Psi(t)\rangle\,|^2 = \frac{2\alpha\omega}{3} \frac{(\hbar\omega)^2}{mc^2 E_{\text{R}}} |r_{ab}/a_{\text{B}}|^2,$$

where E_{R} is the Rydberg energy, a_{B} the Bohr radius and ω is the frequency resonant to the transition $E_n \to E_{n'}$. Further, the dipole matrix element $\boldsymbol{r}_{ab}$ is given by

$$\boldsymbol{r}_{ab} = \langle n', l', m' |\, \boldsymbol{r}\, | n, l, m\rangle \,. \tag{6}$$

a) First rewrite $\boldsymbol{r} = (x, y, z)^T = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$. Then express this vector in terms of the spherical harmonics $Y_{l,m}(\theta, \phi)$. Use this to find the transitions which are allowed in first order. **1**pt(s)

b) Explicitly calculate the dipole matrix elements for the $n = 2 \to n = 1$ transition. **1**pt(s)

c) For an Hydrogen atom prepared in any of the $n = 2$ states, what are the average life times? **1**pt(s)

Problem 9.3: Stark Effect for a Harmonic Oscillator [**Oral** | 2 pt(s)]

ID: ex_stark_effect_harmonic_oscillator:fqt2526

Learning objective

The goal of this problem is to apply a static perturbation (as derived in the lecture) up to second order for a simple setup. This setup has the special property that an exact solution exists and therefore the perturbation theory can be tested by a comparison with the exact solution.

The Hamiltonian of a one-dimensional harmonic oscillator within a homogeneous electric field E is given by

$$H = \frac{1}{2}\left(P^2 + Q^2\right) + eEQ\,, \tag{7}$$

where the Hamiltonian is dimensionless, i.e., $[Q, P] = i$. Consider the second term of the Hamiltonian as a perturbation of the free oscillator, that is, $H_1 = Q$ and $\lambda = eE$.

a) Calculate the perturbed eigenfunctions and energy eigenvalues up to second order in λ. **1**pt(s)

b) Compare the result of perturbation theory with the exact solution for the energy of the problem. **1**pt(s)