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## Problem 11.1: Properties of bosonic operators

[**Oral** | 10 pt(s)]

ID: ex_properties_of_bosonic_operators:fqt2425

**Learning objective**

The first part of this problem reviews properties of bosonic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

In order to solve the harmonic oscillator, one can introduce bosonic creation and annihilation operators satisfying the commutation relation

$$[b, b^\dagger] = 1 \,. \tag{1}$$

The occupation number operator is given by $\hat{n} = b^\dagger b$ with eigenstates $|n\rangle$ and eigenvalues $n$.

a) Using (1), show that $b\,|n\rangle$ and $b^\dagger\,|n\rangle$ are eigenstates of $\hat{n}$. **2**pt(s)

b) Prove the following relations: **2**pt(s)

$$b\,|n\rangle = \sqrt{n}\,|n-1\rangle \ , \tag{2}$$

$$b^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \ . \tag{3}$$

c) Show that there has to be a state $|G\rangle$ with $b\,|G\rangle = 0$ and prove that $n$ is an integer. **2**pt(s)
**Hint:** Use the fact that there are no states with negative norm.

d) Using the commutation relations of operators in bosonic Fock space, **2**pt(s)

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0 \tag{4}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} \ldots \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}}\,|0\rangle \,, \tag{5}$$

prove the following relations:

$$b_i\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle \ , \tag{6}$$

$$b_i^\dagger\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i+1}\,|n_1, \ldots, n_i + 1, \ldots\rangle \ . \tag{7}$$

e) Show that the states (5) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum b_i^\dagger b_i$. Now take a Hilbert-space of $m$-modes, i.e., the basis states of the Fock space are **2**pt(s)

$$|n_1, \ldots, n_m\rangle \,.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number $N\,|\Psi\rangle = \hat{n}\,|\Psi\rangle$?

## Problem 11.2: Properties of fermionic operators

[**Written** | 10 pt(s)]

ID: ex_properties_of_fermionic_operators:fqt2425

**Learning objective**

This problem reviews properties of fermionic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

Now we introduce fermionic operators, which fulfill the anti-commutation relation

$$\{a, a^\dagger\} = 1\,. \tag{8}$$

The occupation number operator is given by $\hat{n} = a^\dagger a$ with eigenstates $|n\rangle$ and eigenvalues $n$.

a) Using (8), show that $a\,|n\rangle$ and $a^\dagger\,|n\rangle$ are eigenstates of $\hat{n}$. **2**pt(s)

b) Prove the following relations: **2**pt(s)

$$a\,|n\rangle = \sqrt{n}\,|1-n\rangle \qquad\qquad a^\dagger\,|n\rangle = \sqrt{1-n}\,|1-n\rangle\,. \tag{9}$$

c) Show that there has to be a state $|G\rangle$ with $a\,|G\rangle = 0$ and a state $|H\rangle$ with $a^\dagger\,|H\rangle = 0$. Further show that these are the only states in the Hilbert space. Assume that $n$ is an integer. **2**pt(s)
**Hint:** Use the fact that there are no states with negative norm.

d) Using the anti-commutation relations of operators in fermionic Fock space, **2**pt(s)

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0 \tag{10}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots (a_i^\dagger)^{n_i} \ldots (a_1^\dagger)^{n_1}\,|0\rangle\,, \tag{11}$$

prove the following relations:

$$a_i\,|n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,1}(-1)^{S_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle\,, \tag{12}$$

$$a_i^\dagger\,|n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,0}(-1)^{S_i}\,|n_1, \ldots, n_i + 1, \ldots\rangle\,, \tag{13}$$

where $S_i = n_\infty + \cdots + n_{i+1}$

e) Show that the states (11) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum a_i^\dagger a_i$. Now take a Hilbert-space of $m$-modes, i.e., the basis states of the Fock space are **2**pt(s)

$$|n_1, \ldots, n_m\rangle\,.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number $N\,|\Psi\rangle = \hat{n}\,|\Psi\rangle$?

## Problem 11.3: Expectation values of bosonic and fermionic operators [**Oral** | 6 pt(s)]

ID: ex_expectation_values_fock_space:fqt2425

**Learning objective**

Here you will take the basis states of the bosonic and fermionic Fock space and practice the calculation of expectation values of many-particle systems.

In the following, we will work with the operators

$$A = c_i^\dagger c_i \qquad B = c_i^\dagger c_i c_j^\dagger c_j$$
$$C = c_i^\dagger c_j^\dagger c_j c_i \qquad D = c_i^\dagger c_j^\dagger c_i c_j .$$

a) Show that these operators are self-adjoint. Consider the cases $c_i = b_i$ (bosons) and $c_i = a_i$ (fermions). **2**pt(s)

b) Calculate the expectation value of the operators $A$, $B$, $C$ and $D$, taking the states (5), with $c_i = b_i$, and (11), with $c_i = a_i$. **2**pt(s)

c) Finally, determine the matrix element **2**pt(s)

$$\langle m_1, \ldots, m_i, \ldots | \, c_i^\dagger c_j + c_j^\dagger c_i \, | n_1, \ldots, n_i, \ldots \rangle ,$$

again both for the bosonic and fermionic Fock space and operator algebra.