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Problem 10.1: Particles in a Well

[**Oral** | 6 pt(s)]

ID: ex_in_distinguishabe_particles_in_1d_potential_well:fqt2425

> **Learning objective**
>
> In this exercise, we will exploit the symmetrization of the wave functions for determining the ground state and excited states of distinguishable and indistingishable particles in a 1D infinite potential well.

Consider a system of three non-interacting particles that are confined to move in a one-dimensional infinite potential well of length a such that $V(x) = 0$ for $0 < x < a$ and $V(x) = \infty$ for other values of x. Determine the energy and wave function of the ground state and the first and second excited states when the three particles are:

a) Spinless and distinguishable with masses $m_1 < m_2 < m_3$. **2**pt(s)

b) Identical bosons. **2**pt(s)

c) Identical spin $\frac{1}{2}$ particles. **2**pt(s)

Problem 10.2: Particles in a harmonic oscillator potential

[**Written** | 4 pt(s)]

ID: ex_2_particles_system_harmonic_oscillator:fqt2425

> **Learning objective**
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> Similarly to the previous exercise, we will exploit the symmetrization of the wave functions for determining the ground state and excited states of a system made of two non-interacting identical particles under the influence of an external harmonic oscillator potential.

Consider a system of two non-interacting identical particles moving in a common external harmonic oscillator potential. Since the particles are non-interacting and identical, their Hamiltonian is given by $\hat{H} = \hat{H}_1 + \hat{H}_2$, where $\hat{H}_1$ and $\hat{H}_2$ are the Hamiltonians of particles 1 and 2:

$$\hat{H}_j = -\left(h^2/2m\right) d^2/dx_j^2 + m\omega x_j^2/2$$

with $j = 1, 2$. The total energy of the system is $E_{n_1 n_2} = \varepsilon_{n_1} + \varepsilon_{n_2}$, where $\varepsilon_{n_j} = \left(n_j + \frac{1}{2}\right) h\omega$.

a) We first consider two spin-1 particles. The spin states corresponding to $S = 2$ are given by **2**pt(s)

$$|2, \pm 2\rangle = |1, 1; \pm 1, \pm 1\rangle, \quad |2, \pm 1\rangle = \frac{1}{\sqrt{2}}(|1, 1; \pm 1, 0\rangle + |1, 1; 0, \pm 1\rangle),$$

$$|2, 0\rangle = \frac{1}{\sqrt{6}}(|1, 1; 1, -1\rangle + 2|1, 1; 0, 0\rangle + |1, 1; -1, 1\rangle),$$

those corresponding to $S = 1$ by

$$|1, \pm 1\rangle = \frac{1}{\sqrt{2}}(\pm|1,1;\pm 1,0\rangle \mp |1,1;0,\pm 1\rangle),$$
$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|1,1;1,-1\rangle - |1,1;-1,1\rangle),$$

and the one corresponding to $S = 0$ by

$$|0,0\rangle = \frac{1}{\sqrt{3}}(|1,1;1,-1\rangle - |1,1;0,0\rangle + |1,1;-1,1\rangle).$$

Determine the energy and the wave functions of the ground state and the first excited state for two spin 1 particles with no orbital angular momentum.

b) Calculate the same quantities for two spin $\frac{1}{2}$ particles. Remember in this case that the singlet (anti-symmetric) and triplet (symmetric) states are given by: **2**pt(s)

$$\chi_{\text{triplet}}\,(\boldsymbol{S}_1, \boldsymbol{S}_2) = \begin{cases} \left|\frac{1}{2}\frac{1}{2}\right\rangle_1 \left|\frac{1}{2}\frac{1}{2}\right\rangle_2, \\ \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\frac{1}{2}\right\rangle_1 \left|\frac{1}{2}-\frac{1}{2}\right\rangle_2 + \left|\frac{1}{2}-\frac{1}{2}\right\rangle_1 \left|\frac{1}{2}\frac{1}{2}\right\rangle_2\right), \\ \left|\frac{1}{2}-\frac{1}{2}\right\rangle_1 \left|\frac{1}{2}-\frac{1}{2}\right\rangle_2, \end{cases}$$

and:

$$\chi_{\text{singlet}}\,(\boldsymbol{S}_1, \boldsymbol{S}_2) = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\frac{1}{2}\right\rangle_1 \left|\frac{1}{2}-\frac{1}{2}\right\rangle_2 - \left|\frac{1}{2}-\frac{1}{2}\right\rangle_1 \left|\frac{1}{2}\frac{1}{2}\right\rangle_2\right).$$