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Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Fortgeschrittene Quantentheorie"***:

- You can find detailed information on lecture and tutorials on the website of our institute:
  `https://itp3.info/fqt2425`
- **Written** problems have to be handed in via ILIAS and will be corrected by the tutors. You must earn at least **50 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **3** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Foundations of Quantum Mechanics

[**Written** | 6 pt(s)]

ID: ex_foundations_of_quantum_mechanics:fqt2425

Learning objective

This problem reviews key concepts of quantum mechanics and its mathematical framework. It is based on the paper *Student understanding of quantum mechanics*, C. Singh, Am. J. Phys. **69**, 885 (2001); `http://dx.doi.org/10.1119/1.1365404`. The main goal is to address and eliminate common misconceptions.

We refer to a generic observable Q and its corresponding quantum mechanical operator $\hat{Q}$. For all of the questions, the Hamiltonian and operators $\hat{Q}$ do not depend on time explicitly.

a) The eigenvalue equation for an operator $\hat{Q}$ is given by $\hat{Q}\,|\psi_i\rangle = \lambda_i\,|\psi_i\rangle$, where $i = 1, \ldots, N$. Write an expression for $\langle\phi|\hat{Q}|\phi\rangle$, where $|\phi\rangle$ is a general state, in terms of the amplitudes $\langle\phi|\psi_i\rangle$. **1**pt(s)

b) If you make measurements of a physical observable Q on a system in immediate succession, do you expect the outcome to be the same every time? Justify your answer. **1**pt(s)

c) If you make measurements of a physical observable Q on an ensemble of identically prepared systems which are not in an eigenstate of $\hat{Q}$, do you expect the outcome to be the same every time? Justify your answer. **1**pt(s)

d) A particle is in a one-dimensional harmonic oscillator potential. Under what conditions will the expectation value of an operator $\hat{Q}$ depend on time if **1**pt(s)

   i. the particle is initially in a momentum eigenstate?

   ii. the particle is initially in an energy eigenstate?

Justify your answer in both cases.

e) Questions (i)-(ix) refer to the following system: An electron is in a uniform magnetic field B which is pointing in the z-direction. The Hamiltonian for the spin-degree of freedom for this system is given by $\hat{H} = -\gamma B \hat{S}_z$, where γ is the gyromagnetic ratio and $\hat{S}_z$ is the z-component of the spin angular momentum operator. **2**pt(s)

Notation: $\hat{S}_z \left|\uparrow\right\rangle = \hbar/2 \left|\uparrow\right\rangle, \hat{S}_z \left|\downarrow\right\rangle = -\hbar/2 \left|\downarrow\right\rangle$.
For reference, the unnormalized eigenstates of $\hat{S}_x$ and $\hat{S}_y$ are given by

$$\hat{S}_x(\left|\uparrow\right\rangle \pm \left|\downarrow\right\rangle) = \pm\hbar/2(\left|\uparrow\right\rangle \pm \left|\downarrow\right\rangle) \tag{1}$$

$$\hat{S}_y(\left|\uparrow\right\rangle \pm i\left|\downarrow\right\rangle) = \pm\hbar/2(\left|\uparrow\right\rangle \pm i\left|\downarrow\right\rangle) \,. \tag{2}$$

i. If you measure S_z of a state $|\chi\rangle = (\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)/\sqrt{2}$, what are the possible results, and what are their respective probabilities?
ii. If the result of the first measurement of S_z was $\hbar/2$, and you immediately measure S_z again, what are the possible results, and what are their respective probabilities?
iii. If the result of the first measurement of S_z was $\hbar/2$, and you immediately measure S_x, what are the possible results, and what are their respective probabilities?
iv. What is the expectation value $\langle\hat{S}_z\rangle$ of the state $|\chi\rangle = (\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)/\sqrt{2}$?
v. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_y$ depend on time? Justify your answer.
vi. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_z$ depend on time? Justify your answer.
vii. If the electron is initially in an eigenstate of $\hat{S}_z$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.
viii. If the electron is initially in an eigenstate of $\hat{S}_z$, does the expectation value of $\hat{S}_z$ depend on time? Justify your answer.
ix. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.

Problem 1.2: Spin $s = 1/2$ particles

[**Oral** | 4 pt(s)]

ID: ex_pauli_matrices_aqm:fqt2425

Learning objective

This exercise aims to review some properties of Pauli matrices for spin $s = 1/2$ particles while providing an opportunity to familiarize oneself with algebra involving Levi-Civita and Kronecker Delta symbols, which are commonly used in Quantum Mechanics.

For a particle with spin $s = 1/2$, the Pauli matrices can be defined as

$$\hat{\sigma}_1 = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}_3 = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

with the following multiplication rule

$$\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbb{I}_2 + i\varepsilon_{ijk}\hat{\sigma}_k \quad \text{with} \quad i,j,k = 1,2,3. \tag{4}$$

The Levi-Civita symbol in a three-dimensional space is represented as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2)\,, \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3)\,, \\ 0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i, \end{cases} \tag{5}$$

i.e., it takes values +1 (-1) for even (odd) permutations of the indices $(i,j,k) = (1,2,3)$ or 0 if any index is repeated. The Kronecker Delta is simply defined as

$$\delta_{ij} \equiv \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases} \tag{6}$$

a) Using property (4) prove that **1pt(s)**

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\,(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b}\mathbb{I}_2 + i\,(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma} \quad \text{where} \quad \mathbf{a},\mathbf{b}\in\mathbb{R}^3. \tag{7}$$

b) Show that (i) $[\mathbf{a}\cdot\boldsymbol{\sigma},\ \mathbf{b}\cdot\boldsymbol{\sigma}] = 2\,i\,(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}$ and (ii) $\mathrm{Tr}(\mathbf{a}\cdot\boldsymbol{\sigma}\ \mathbf{b}\cdot\boldsymbol{\sigma}) = 2\ \mathbf{a}\cdot\mathbf{b}$ with equation (7). **2pt(s)**

c) Find the eigenvalues and normalized eigenfunctions of the operators $\hat{S}_i = \hbar\hat{\sigma}_i/2$ for $i = x,y,z$. **1pt(s)**

Problem 1.3: Commutators

[**Oral** | 5 pt(s)]

ID: ex_commutators_2:fqt2425

Learning objective

We review some important commutation relations with focus on the position $\hat{x}$, linear momentum $\hat{p}$, and angular momentum $\hat{L}$ operators.

a) Show that the following identity holds for commutators of products **1pt(s)**

$$[A,BC] = B[A,C] + [A,B]C\,. \tag{8}$$

b) Let $g(\hat{x})$ and $f(\hat{p})$ be analytical functions of the position and momentum operators, respectively. Show that **1pt(s)**

$$[\hat{p}, g(\hat{x})] = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}g(\hat{x})\,, \tag{9}$$

$$[\hat{x}, f(\hat{p})] = +i\hbar\frac{\mathrm{d}}{\mathrm{d}p}f(\hat{p})\,. \tag{10}$$

c) Using the previous exercises, calculate the following commutators involving the canonical position and momentum operators $\hat{x}$ and $\hat{p}$ **1pt(s)**

$$\left[\hat{x},\hat{p}^2\right], \qquad \left[\hat{x}^2,\hat{p}^2\right], \qquad \left[\hat{x}\hat{p},\hat{p}^2\right]. \tag{11}$$

d) Using just the canonical commutation relations $[\hat{x}_l,\hat{p}_m] = i\hbar\delta_{lm}$, calculate $[\hat{L}_\alpha,\hat{L}_\beta]$, where $\hat{L}_\alpha = \varepsilon_{\alpha ij}\hat{x}_i\hat{p}_j$ is the α-component of the angular momentum operator. **1pt(s)**

e) Now calculate $[\hat{L}_\alpha, \hat{\boldsymbol{L}}^2]$ and $[\hat{L}_\alpha,\hat{p}^2]$. **1pt(s)**