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May 11th, 2026
SS 2026

Problem 5.1: Electric field of a dipole

[**Oral** | 6 pt(s)]

ID: ex_electric_field_dipole:edyn26

Learning objective

In the first part of the problem, we calculate the electric field for a dipole. The resulting expression contains a δ-function term, whose physical importance is discussed in the second part of the problem.

a) Recall the important result $\Delta\frac{1}{|\boldsymbol{r}|} = -4\pi\delta^3(\boldsymbol{r})$ from Problem 2.1 and generalize it to **1**pt(s)

$$\partial_\alpha\partial_\beta\frac{1}{|\boldsymbol{r}|} = -\frac{\delta_{\alpha\beta}}{|\boldsymbol{r}|^3} + 3\frac{x_\alpha x_\beta}{|\boldsymbol{r}|^5} - \frac{4\pi}{3}\delta_{\alpha\beta}\,\delta^3(\boldsymbol{r}). \tag{1}$$

Hint: Use a symmetry argument and the result from exercise Problem 2.1 to derive the last term in equation (1).

b) The electric potential for a point dipole $\boldsymbol{p}$ is given by $\phi(\boldsymbol{r}) = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0|\boldsymbol{r}|^3} = -(\boldsymbol{p}\cdot\boldsymbol{\nabla})\frac{1}{4\pi\epsilon_0|\boldsymbol{r}|}$. Using relation (1), show that the electric field of the dipole can be written as ($\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$): **1**pt(s)

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\hat{\boldsymbol{r}}\cdot\boldsymbol{p})\,\hat{\boldsymbol{r}} - \boldsymbol{p}}{|\boldsymbol{r}|^3} - \frac{4\pi}{3}\boldsymbol{p}\,\delta^3(\boldsymbol{r})\right]. \tag{2}$$

The δ-function term in equation (2) is a correction for $\boldsymbol{r} = 0$. In the following, we are going to re-derive it in a different way to understand its physical origin.

We would like to prove the following THEOREM: The *average electric field* over the volume V enclosed by a sphere of radius R, due to an arbitrary charge distribution within the sphere, is given by

$$\overline{\boldsymbol{E}} = -\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}}{R^3}, \tag{3}$$

where $\boldsymbol{p}$ is the total dipole moment with respect to the center of the sphere.

c) To do this, first calculate the average electric field within the sphere (with enclosed volume V), due to a single charge q at position $\boldsymbol{r}_q$: **1**pt(s)

$$\overline{\boldsymbol{E}}_q = \frac{1}{V}\int_V \mathrm{d}^3r\,\boldsymbol{E}_q(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{V}\int_V \mathrm{d}^3r\,\frac{\boldsymbol{r}-\boldsymbol{r}_q}{|\boldsymbol{r}-\boldsymbol{r}_q|^3}. \tag{4}$$

Realize that this expression can also be considered as the electric field *at the position* $\boldsymbol{r}_q$, that is generated by a (fictional) ball with a uniform charge density $\rho = -q/V$. Use this analogy to calculate $\overline{\boldsymbol{E}}_q$ via Gauss's law.

d) Use the superposition principle to generalize the result for the point charge q to arbitrary charge distributions and prove equation (3). **1**pt(s)

e) Explicitly calculate the average electric field that is generated by a point-like dipole, by integrating the electric field from equation (2) over a ball. In your integration, start by excluding a small region around the origin. 1pt(s)

f) Finally, show that the δ-function term in equation (3) is essential to satisfy the average-value theorem. 1pt(s)

Note: Another approach is to calculate the electric field of a homogeneously polarized ball of radius a. Outside of the ball, the field is exactly given by equation (2). Inside the ball, the field has a constant value $\boldsymbol{E}_{\text{in}} = -1/4\pi\epsilon_0 \cdot \boldsymbol{p}/a^3$, where $\boldsymbol{p}$ is the dipole moment of the ball. As the size of the ball goes to zero, the field strength goes to infinity in such a way that the integral over the ball remains constant, giving the prefactor of the δ-function: $-\boldsymbol{p}/3\epsilon_0$.

Problem 5.2: Spherical multipole moment [**Oral** | 2 pt(s)]

ID: ex_spherical_multipole_moment:edyn26

Learning objective

The goal of this problem is to calculate the spherical multipole moments q_{lm} for different charge distributions and to study when a quadrupole moment occurs.

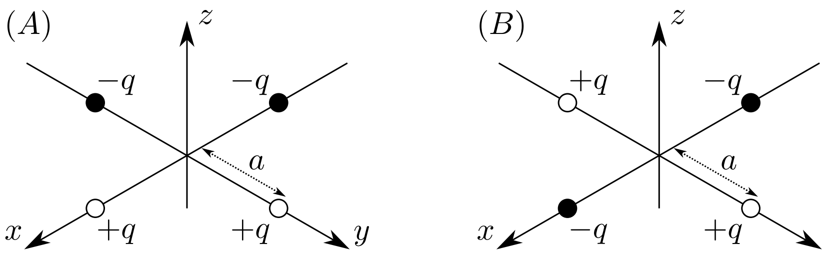
Let us consider a localized distribution of charge described by a charge density $\rho(\boldsymbol{r})$, which is non-vanishing only inside a sphere of radius R^* around the origin. The potential outside the sphere can be written as an expansion in spherical harmonics $Y_{lm}(\theta, \phi)$ as

$$\phi(\boldsymbol{r}) = k \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi), \tag{5}$$

where the spherical multipole moments q_{lm} are given by

$$q_{lm} = \int \mathrm{d}^3\boldsymbol{r}\, Y^*_{lm}(\theta, \phi) r^l \rho(\boldsymbol{r}). \tag{6}$$

Here, we perform calculations for two charge distributions (A) and (B). Both consist of four charges in the xy-plane, placed distance a from the origin and equidistant to each other. The distributions are given in the sketch



a) Write down the charge distribution in spherical coordinates for the two charge distributions of interest. 1pt(s)

b) Compute the spherical monopole, dipole and quadrupole moments for both arrangements. 1pt(s)

Problem 5.3: Method of images: electric dipole near a conducting sphere [**Written** | 2 pt(s)]

ID: ex_method_images_dipole_sphere:edyn26

Learning objective

In this exercise, we will apply the method of images to find the electrostatic potential and the interaction energy between an electric dipole and a conducting, grounded sphere.

Consider a conducting, grounded sphere of radius R located at the origin and an electric dipole $\boldsymbol{p}$ at position $\boldsymbol{r}$ ($|\boldsymbol{r}| > R$).

Hint: It is useful to represent the dipole $\boldsymbol{p}$ at position $\boldsymbol{r}$ as a pair of point charges $+q$ and $-q$ located at $\boldsymbol{r} \pm \boldsymbol{p}h/(2|\boldsymbol{p}|)$, respectively, then apply the method of images to each charge and take the limit $h \to 0$, while keeping $qh = |\boldsymbol{p}|$ fixed.

a) Find the electrostatic potential in the system for an arbitrary orientation of the dipole. **1**pt(s)

b) Calculate the interaction energy between the dipole and the induced charges on the sphere. **1**pt(s)