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April 21st, 2026
SS 2026

Problem 2.1: Green's function for the Poisson equation

[**Oral** | 3 pt(s)]

ID: ex_greens_function_poisson_equation:edyn26

Learning objective

In this problem, we prove the important identity of electrostatic Green's function in two ways. First with Gauss theorem and second by taking Fourier transformation of Yukawa potential in the zero mass limit.

In the lectures, you have discussed Green's functions—a mathematical tool to solve inhomogeneous differential equations. Let $\hat{L}(\boldsymbol{x})$ be a linear differential operator. The Green's function $G(\boldsymbol{x}, \boldsymbol{x}')$ is defined as the solution to the equation

$$\hat{L}(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{1}$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}')$ is the Dirac delta function. Once the Green's function is known, the solution to the inhomogeneous equation $\hat{L}(\boldsymbol{x})u(\boldsymbol{x}) = f(\boldsymbol{x})$, where $f(\boldsymbol{x})$ is a source term, can be expressed as

$$u(\boldsymbol{x}) = \int G(\boldsymbol{x}, \boldsymbol{x}')f(\boldsymbol{x}')\, d\boldsymbol{x}'. \tag{2}$$

The Dirac delta function can be heuristically defined as follows

$$\delta(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \neq 0, \quad \int \delta(\boldsymbol{x})\, d\boldsymbol{x} = 1. \tag{3}$$

It also satisfies the following property for the well-behaved function $f(\boldsymbol{x})$:

$$\int \delta(\boldsymbol{x} - \boldsymbol{x}')f(\boldsymbol{x}')\, d\boldsymbol{x}' = f(\boldsymbol{x}). \tag{4}$$

Remark: The delta function is not a function in a "classical" sense. It can be rigorously defined as a distribution or a generalized function, which is a linear functional that acts on a space of test functions (usually these are functions that have certain smoothness and decay fast enough at the infinity). For the introduction to the theory of distributions and Green's functions, see, e.g., lecture notes [clickable].

Consider the following important identity

$$\Delta \frac{1}{|\boldsymbol{r}|} = -4\pi\, \delta^3(\boldsymbol{r}). \tag{5}$$

a) Show that $\Delta \frac{1}{|\boldsymbol{r}|} = 0$ for any $\boldsymbol{r} \neq 0$. Then consider a ball with a radius R centered at the origin, which we denote by B_R. Since the domain of $\Delta|\boldsymbol{r}|^{-1}$ is $\mathbb{R}^3 \setminus \{0\}$, we cannot directly integrate 1pt(s) it over the ball B_R. However, in the distributional sense, we can formally define the volume integral of $\Delta|\boldsymbol{r}|^{-1}$ via the surface integral (Gauss' theorem) as

$$\int_{B_R} \Delta \frac{1}{|\boldsymbol{r}|}\, d^3r \stackrel{\text{def}}{=} \int_{\partial B_R} \boldsymbol{\nabla} \frac{1}{|\boldsymbol{r}|} \cdot d\boldsymbol{S}. \tag{6}$$

Show that the surface integral equals to -4π for any $R > 0$, which is consistent with the identity in equation (5).

The Yukawa potential is given by the expression

$$\phi_Y(\boldsymbol{r}) = \frac{\mathrm{e}^{-mr}}{r} \tag{7}$$

where $r = |\boldsymbol{r}|$ and $m > 0$ is the mass of the particle that mediates the potential. The inverse mass is propotional to a length scale (Compton length) that determines the range of the potential. If photons had a rest mass, the Coulomb potential would have to be replaced by the Yukawa potential. We can see that the Coulomb potential is the limiting case of $\phi_Y(\boldsymbol{r})$ in the zero-mass limit (infinite-range limit).

b) Use the three-dimensional Fourier transformation of $\Delta\frac{1}{|\boldsymbol{r}|}$ to show that equation (5) holds. First calculate the Fourier transform of $\frac{1}{|\boldsymbol{r}|}$ by transforming the Yukawa potential and then take the zero-mass limit. **1**pt(s)

c) Calculate the Fourier transformation of the equation **1**pt(s)

$$\left[\Delta - m^2\right]\phi_Y(\boldsymbol{r}) = -4\pi\, \delta^3(\boldsymbol{r}), \tag{8}$$

and evaluate the solution in Fourier space for $\phi_Y(\boldsymbol{k})$. Determine $\phi_Y(\boldsymbol{r})$ by explicitly evaluating the Fourier transformation and demonstrate that the solution is the Yukawa potential. **Hint:** Use the result from (b) to obtain the solution in Fourier space. The Fourier transformation into real space can be achieved using the residue theorem.

Problem 2.2: Cavendish experiment, part 1: Spherical capacitor [**Oral** | 3 pt(s)]

ID: ex_cavendish_experiment_part1:edyn26

> **Learning objective**
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> In this problem, we will learn to calculate the electric field, scalar potential and capacitance of a spherical capacitor which is made of two concentric metallic spherical shells.

A spherical capacitor is given by two concentric, metallic spherical shells with radii $R_1 < R_2$ and respective charges $Q_1 = Q$ and $Q_2 = -Q$.

a) Calculate the electric field $\boldsymbol{E}(\boldsymbol{r})$ of the given arrangement for the three regions $r < R_1$, $R_1 < r < R_2$ and $R_2 < r$. **1** pt(s)

b) Determine the scalar potential $\phi(\boldsymbol{r})$ with the boundary condition $\phi(\boldsymbol{r}) \longrightarrow 0$ for $r \longrightarrow \infty$ such that it is continuous at R_1 and R_2. **1** pt(s)

c) Determine the capacitance of the spherical capacitor. **1** pt(s)

Problem 2.3: Cavendish experiment, part 2 [**Written** | 2 (+1 bonus) pt(s)]

ID: ex_cavendish_experiment_part2:edyn26

> **Learning objective**
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> In this problem, we will calculate the total energy of the spherical capacitor by calculating the electrostatic interaction energy. We also calculate the charge distribution on both spheres.

In 1772, Cavendish designed the following experiment to verify Coulombs law. He used a spherical capacitor with the two spheres initially connected by an electrical contact. He then placed a static charge on the outer sphere and subsequently removed the contact between the spheres. After removing the outer sphere, he confirmed that the inner sphere had not been charged. This is a special property of the $1/r$ Coulomb law where the electric potential inside a conducting sphere is constant (the electric field vanishes).

a) Assume that the electrostatic potential is instead given by the modified power law $\phi_\epsilon(\boldsymbol{r}) = \frac{1}{|\boldsymbol{r}|^{1-\epsilon}}$. Determine the charge on the inner sphere for both prospective potentials as follows. Let V_i be the "volume" of the ith sphere. We can calculate the electrostatic interaction energy by **1** pt(s)

$$E_{ij} = \frac{1}{2} \int\limits_{V_i} \mathrm{d}^3 r \int\limits_{V_j} \mathrm{d}^3 r' \, \rho(\boldsymbol{r}) \rho(\boldsymbol{r}') \phi(\boldsymbol{r} - \boldsymbol{r}'), \tag{9}$$

where E_{ii} is the self energy of the ith sphere and $E_{12} + E_{21} = 2E_{12}$ is the interaction energy between the two spheres. The full energy is given by $E = E_{11} + E_{22} + 2E_{12}$.

i) Due to the spherical symmetry, the charge density will be homogeneously distributed on the spheres. Assume that the spheres have a surface density of $\sigma_i = Q_i/S_i$ were S_i is the surface area of the respective sphere. Calculate the total energy $E(Q_1, Q_2)$ for arbitrary charges.

ii) Determine the actual charge distribution by minimizing the energy with the constraint that the total charge $Q = Q_1 + Q_2$ is fixed.

b) Repeat the analysis assuming that electrostatic potential is given by the Yukawa potential with a finite photon mass. **1**pt(s)

*c) How can we derive the capacitance from Problem 2.2c from the energy function $E(Q_1, Q_2)$ for the Coulomb potential (set $\epsilon = 0$ or $m = 0$ in your solution). **+1**pt(s)