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Problem 8.1: Green's function for the wave equation

[**Written** | 4 pt(s)]

ID: ex_greens_function_wave_equation:edyn24

> **Learning objective**
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> In the first exercise you will calculate the Green's function of the wave equation. In frequency and momentum space this Green's function possesses poles, whose location have physical relevance for causality. We verify our solution by obtaining the Green's function in a second way.

In the lecture, we have seen that the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ satisfies the following wave equation within Coulomb gauge:

$$\left[\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\boldsymbol{A}(\boldsymbol{r}, t) = -\mu_0\,\boldsymbol{j}_{\mathrm{t}}(\boldsymbol{r}, t). \tag{1}$$

Here, $\boldsymbol{j}_{\mathrm{t}} = \boldsymbol{j}(\boldsymbol{r}, t) - \epsilon_0 \nabla \partial_t \phi(\boldsymbol{r}, t)$ is the transversal part of the current density: We are going to solve Eq. (1) with the Green's function method. Let $G(\boldsymbol{r}, t; \boldsymbol{r}', t')$ be the Green's function for the operator $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$, i.e.

$$\Box\, G(\boldsymbol{r}, t; \boldsymbol{r}', t') = 4\pi\delta^3(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'). \tag{2}$$

a) Write the general solution of Eq. (1) in terms of the Green's function. **1**pt(s)

b) Show that the Fourier transform of the Green's function, for $\boldsymbol{r}' = 0$ and $t' = 0$, is given by **1**pt(s)

$$G(\boldsymbol{k}, \omega) = \frac{4\pi}{k^2 - \omega^2/c^2}. \tag{3}$$

c) The Fourier transformed $G(\boldsymbol{k}, \omega)$ has poles at $k = |\boldsymbol{k}| = \pm\omega/c$. Calculate the Green's function $G^{\mathrm{r,a}}(\boldsymbol{r}, t; \boldsymbol{r}', t')$ by inverse Fourier transformation. The indices 'r' and 'a' stand for the retarded and the advanced Green's function, respectively. The two functions are distinguished by the way the two poles are handled. **1**pt(s)

   Transform from $\boldsymbol{k}$-space to real-space first. Use spherical coordinates and rewrite the integral over $k \in [0, \infty)$ into an integral over $k \in (-\infty, \infty)$. In which way can you close the integration contour in the complex k plane? Shift the poles infinitesimally from the real axis by setting $\omega/c \longrightarrow \omega/c \pm i\epsilon$ (how does this affect the position of the poles?). Then, use the residue theorem to solve the k integration. Finally, perform the transformation from the frequency domain back to the time domain.

   The Green's function is called retarded (advanced) if both poles of $G(\boldsymbol{k}, \omega)$ as a function of ω lie in the lower (upper) half plane.

d) To see, that the previous calculations are correct, we can calculate the Green's function in a different way. To do this, first Fourier transform Eq. (2) ($\boldsymbol{r}' = 0$ and $t' = 0$) in time to get 1pt(s)

$$\left[-\left(\frac{\omega}{c}\right)^2 - \nabla^2\right] G(\boldsymbol{r}, \omega) = 4\pi\delta(\boldsymbol{r}). \tag{4}$$

Next, show that at any point $r \neq 0$ the ansatz

$$G(\boldsymbol{r}, \omega) = A\frac{e^{i\frac{\omega}{c}|\boldsymbol{r}|}}{|\boldsymbol{r}|} + B\frac{e^{-i\frac{\omega}{c}|\boldsymbol{r}|}}{|\boldsymbol{r}|} \tag{5}$$

satisfies this differential equation. Using this ansatz, integrate Eq. (4) over a small sphere around the origin to derive a constraint for the constants A and B such that the δ-distribution is fulfilled. Finally, perform the inverse Fourier transform to find $G(\boldsymbol{r}, t)$. Causality requires, that $G(\boldsymbol{r}, t < 0) = 0$ for the retarded Green's function. Use this condition to determine A and B.

Problem 8.2: Wave in a medium [**Oral** | 3 pt(s)]

ID: ex_wave_medium:edyn24

Learning objective

In this exercise you will familiarize yourself with electromagnetic waves in the presence of dielectric media. Especially, you will once again see that the interior of a perfect conductor is free of electrical fields.

Consider a plane wave in a non-conductive medium (in which the conductivity σ disappears and ϵ and μ are constant).

a) Derive the wave equation for the electromagnetic field directly from the Maxwell equations. Show that the plane wave 1pt(s)

$$\begin{pmatrix} \boldsymbol{E}(\boldsymbol{x}, t) \\ \boldsymbol{B}(\boldsymbol{x}, t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{B}_0 \end{pmatrix} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{6}$$

is a solution of these equations and determine the dispersion relation. At what speed does the wave propagate in the medium and at what speed in the vacuum?

Next, we analyze the propagation of light in a metal.

b) Derive the dielectric function $\epsilon(\omega)$ (also called dielectric permittivity) of a conductor in SI units: 1pt(s)

$$\epsilon(\omega) = \epsilon_0 + \frac{i\sigma(\omega)}{\omega}. \tag{7}$$

In the following, use the Drude-conductivity

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}. \tag{8}$$

c) What does the wave equation for the electromagnetic field look like? Show that the plane wave which penetrates the conductive material is attenuated. Calculate the penetration depth δ for a monochromatic wave in the limit of low frequencies. What conclusions can be drawn from this result? **1** pt(s)
Hint: The penetration depth δ is defined as the distance at which the initial wave is attenuated by e^{-1}.

Problem 8.3: Interplanetary sailing

[**Oral** | 2 pt(s)]

ID: ex_interplanetary_sailing:edyn24

Learning objective

A solar sail is a hypothetical propulsion system for space travel. A solar sail uses the light pressure of the sun to gain acceleration. In this exercise we study the feasibility of solar sail for accelerating a probe in earth's orbit.

A planar electromagnetic wave, which spreads in the vacuum, reaches a perfectly conducting flat screen (later called "solar sail") perpendicularly. The energy flux density (energy per unit area, per unit time) transported by the electromagnetic fields is given by the Poynting vector

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}), \tag{9}$$

whereas the momentum density $\mathfrak{P}$, stored in the fields, is $\mathfrak{P} = \boldsymbol{S}/c^2$.

a) Using the conservation of momentum, show that the pressure P applied to the screen (the so-called radiation pressure) is equal to the energy density of the wave. For this purpose, time average the Poynting vector. Comment on why such an averaging is physically justified? **1** pt(s)

b) In the earth's neighborhood the electromagnetic energy flow (originating from the sun) is about $0.14\,\mathrm{W/cm^2}$. What would be the acceleration (caused by the solar radiation pressure) of the spacecraft consisting of a capsule with mass 10^5kg and a "solar sail" with surface density $10^{-4}\,\mathrm{g/cm^2}$ and dimensions $10\,\mathrm{km} \cdot 10\,\mathrm{km}$? **1** pt(s)