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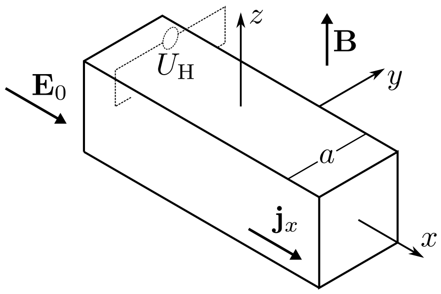
Problem 7.1: Hall Effect

[**Oral** | 4 pt(s)]

ID: ex_hall_effect:edyn24

> **Learning objective**
>
> The Hall effect describes the physics of moving charged particles in materials when exposed to an external magnetic field. In the first part of the problem, we will calculate the Hall voltage and derive the Drude model for transport in materials. In the last part, we will calculate the conductance tensor as well as the Hall coefficient and learn about the importance of the Hall coefficients.



Consider an infinitely long conductor in x-direction with finite extent a in y- and z-direction. An electric field $\boldsymbol{E_0} = E_x \boldsymbol{e}_x$ in x-direction is applied to the conductor, leading to a current density $\boldsymbol{j}_x$. An additional magnetic field $\boldsymbol{B} = B_z \boldsymbol{e}_z$ in z-direction will deflect charge carriers (electrons with charge $-e$) through the Lorentz force. This leads to an accumulation of charges on the sides of the material, resulting in an electric field $\boldsymbol{E}_{\mathrm{H}} = E_y\, \boldsymbol{e}_y$ along the y-direction. This phenomenon is called *Hall effect*.

a) Consider the stationary case ($j_y = v_y = 0$) and calculate the Hall field $\boldsymbol{E}_{\mathrm{H}}$ as well as the potential difference U_{H} between both sides in terms of the current density j_x. In which direction does $\boldsymbol{E}_{\mathrm{H}}$ point? **1**pt(s)
Hint: : The current density is given by $\boldsymbol{j} = -en\boldsymbol{v}$, where n is the electron density and $\boldsymbol{v}$ their velocity. In the stationary case, there is no net force on the carriers in y-direction.

b) The Drude model for transport in metals assumes the following equation of motion for the charge carriers: **1**pt(s)

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = -e\left(\boldsymbol{E} + \frac{1}{m}\boldsymbol{p} \times \boldsymbol{B}\right) - \frac{\boldsymbol{p}}{\tau}. \tag{1}$$

Here, $\boldsymbol{p}$ is the momentum of the electrons, m is the electron mass and τ is the relaxation time (scattering time). Motivate/derive this equation.
Hint: The Drude model assumes that electrons undergo a scattering event with probability $\mathrm{d}t/\tau$ within an infinitesimal short time span $\mathrm{d}t$.

c) Determine the conductance tensor σ_{kl}, which is defined via: (Einstein convention) **1**pt(s)

$$j_k = \sigma_{kl} E_l, \qquad k, l \in \{x, y\} \tag{2}$$

as well as the resistance tensor $\rho = \sigma^{-1}$ within the Drude model.
Hint: Consider the stationary case ($\dot{\boldsymbol{p}} = 0$). Adopt the notation $\sigma_0 = ne^2\tau/m$ for the Drude DC conductance and $\omega_c = eB/m$ for the cyclotron frequency.

d) Calculate the Hall coefficient $R_{\mathrm{H}} = E_y/(j_x B_z)$ for the stationary case within the Drude model. What is the sign of R_{H}? Is it possible to use the Hall effect to determine the charge of (unknown) particles/carriers of the electric current? **1**pt(s)

Problem 7.2: Gauge invariance of the classical equations of motion [**Written** | 1 pt(s)]

ID: ex_gauge_invariance_of_the_classical_equations_of_motion:edyn24

Learning objective

In this problem, we show the gauge invariance of the equations of motion for a charged particle. From the Lagrangian we will derive the Hamiltonian by computing the canonical momentum.

The Lagrange function for a (non-relativistic) charged particle in an electromagnetic field is given by

$$\mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - q\,\phi(\boldsymbol{r}, t) + q\,\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}, t). \tag{3}$$

Derive the equations of motion and show that they are gauge invariant (eichinvariant). Calculate the canonical momentum $\boldsymbol{p} = \partial\mathcal{L}/\partial\dot{\boldsymbol{r}}$. Is it gauge invariant? What is the relation between the mechanical and the canonical momentum? Show that the Hamiltonian is given by

$$\mathcal{H}(\boldsymbol{r}, \boldsymbol{p}, t) = \frac{1}{2m} \left(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{r}, t)\right)^2 + q\phi(\boldsymbol{r}, t). \tag{4}$$

Problem 7.3: Demagnetization*

[**Written** | 5 bonuspt(s)]

ID: ex_demagnetization:edyn24

Learning objective

In this problem we study the effect of demagnetization by placing non-spherical objects into an external magnetic field. We are interested in a cigar shaped and a discus shaped object to investigate the influence of the geometry on the solution.

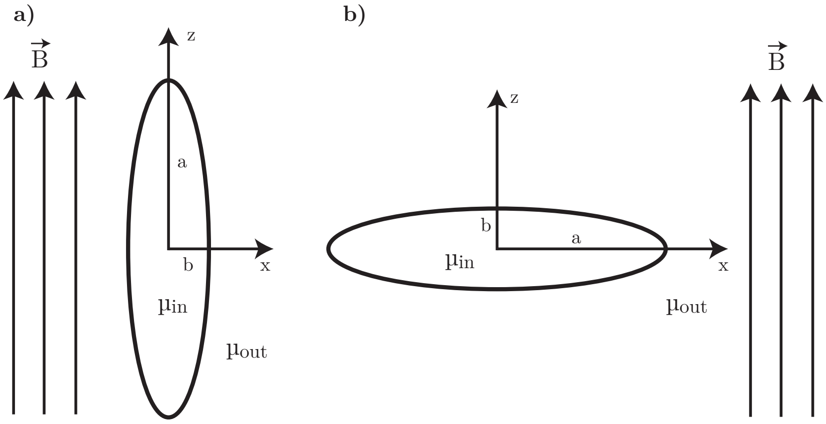


Figure 1: Sketches of the two cases **a)** prolate (cigar shaped) **b)** oblate (discus shaped) object, within a magnetic field.

The rotation axis is the z-axis, the longer (shorter) axis is a (b), see figure. The magnetic field $\boldsymbol{B}$ is parallel to the z-axis. In this scenario we consider the system without currents present, thus $\operatorname{rot}\mathbf{H} = 0$ and $\operatorname{div}\mathbf{H} = 0$. In addition, we are interested in a linear medium, i.e. $\boldsymbol{B} = \mu\,\boldsymbol{H}$. Hence, we can write $\mathbf{H} = -\nabla\Phi$. We label the fields inside the object with $\boldsymbol{H}_{\text{in}}$ and outside with $\boldsymbol{H}_{\text{out}}$.

a) Give the boundary conditions for $\boldsymbol{B}_i$, $\boldsymbol{H}_i$ and Φ_i. Write down the general equations for the surfaces of the given ellipsoids of rotation. **1**pt(s)

b) **Prolate Case:** **2**pt(s)
In order to tackle this problem, we need a handy coordinate system. Here, we choose hyperbolic coordinates with

$$
\begin{aligned}
x &= c\,\sinh u\,\sin v\,\cos\phi \\
y &= c\,\sinh u\,\sin v\,\sin\phi \\
z &= c\,\cosh u\,\cos v. \qquad (5)
\end{aligned}
$$

- What is c? Which values can u, v, ϕ take? How are a and b related to u, v and ϕ on the surface of the ellipsoid.
- Write down the general expression for the Laplace operator Δ in arbitrary orthogonal coordinates.
- Transform the Laplace operator into hyperbolic coordinates.
- Make a separation ansatz in order to solve the Laplace problem $\Delta\Phi = 0$. What are the possible solutions?

- Next, find the solutions for Φ_{in} and Φ_{out} and use the boundary conditions to determine the constants within the ansatz.
- Derive a relation between the fields $\boldsymbol{H}_{\text{in}}$ and $\boldsymbol{H}_{\text{out}}$.

The demagnetization factor is defined through

$$\boldsymbol{H}_{\text{out}} = \left(1 - (1-\mu)\, n\right) \boldsymbol{H}_{\text{in}} \,, \tag{6}$$

where $\mu = \mu_{\text{in}}/\mu_{\text{out}}$.

- Determine n.
- Rearrange this expression to obtain $n = n(\epsilon)$, where ϵ is the eccentricity with $\epsilon = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a}$.
- Investigate $n(\epsilon)$ in the limits $\epsilon \to 0$ and $\epsilon \to 1$. For the latter limit use $\epsilon = \sqrt{1-\eta^2}$ and consider $\eta \to 0$.

Hint: : Everything can be related to Legendre functions.

c) **Oblate Case:** 1pt(s)

We turn our attention to the oblate case and use the coordinates

$$\begin{aligned} x &= c \cosh u \, \sin v \, \cos\phi \\ y &= c \cosh u \, \sin v \, \sin\phi \\ z &= c \sinh u \, \cos v . \end{aligned} \tag{7}$$

Repeat all steps from the previous task for the oblate case and determine n and its limits.

d) Why is n small for a cigar and large for a discus? Give the reasons graphically. Why is the demagnetization factor called like this? 1pt(s)