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May 1st, 2024
SS 2024

Problem 4.1: Cylindrical Capacitor [**Oral** | 1 pt(s)]

ID: ex_cylindrical_capacitor:edyn24

> **Learning objective**
>
> In this exercise, we apply our knowledge to evaluate the capacity of a cylindrical capacitor. Given the translational invariance of the cylinder, we can apply the tools from the previous lecture to find the electric field and charge distribution.

Consider two concentric conducting cylinders of length L with radii $R_1 < R_2$ (coaxial cable) that are separated from each other by an insulating medium such as air or vacuum. Compute the capacitance C per unit length for this setup for the case of very long conductors $L \gg R_2$. How does the capacitance change if the inner cylinder is not hollow, but solid instead ?

Problem 4.2: Legendre polynomials and charge density [**Oral** | 3 pt(s)]

ID: ex_legendre_polynomials_and_charge_density:edyn24

> **Learning objective**
>
> Boundary value problems with a surface given by a sphere can naturally be solved in spherical coordinates. Here, we apply this method to solve the electrostatic problem with Dirichlet boundary conditions on a sphere, and get acquainted with Legendre Polynomials.

Let a spherical shell with radius R carry a fixed charge density $\sigma(\theta)$ such that the potential on the sphere is

$$V(r = R, \theta) = V_0 + V_1 \cos\theta + V_2 \cos 2\theta \tag{1}$$

where V_0, V_1, and V_2 are constants and θ is the polar angle.

a) Find the potential $V(r, \theta)$ inside and outside of the spherical shell ($V(\infty) = 0$). Use the ansatz **1**pt(s)

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} (a_l r^l + b_l r^{-(l+1)}) P_l(\cos\theta), \tag{2}$$

where $P_l(x)$ are the Legendre polynomials.

b) From the potential $V(r, \theta)$ compute the electric field $\boldsymbol{E}(r, \theta)$. **1**pt(s)

c) Oberserve how - as is to be expected - the component of $\boldsymbol{E}$ perpendicular to the spherical shell is discontinuous at $r = R$ with a jump of $\boldsymbol{E}_\perp^{\text{outside}} - \boldsymbol{E}_\perp^{\text{inside}} = \frac{\sigma}{\epsilon_0}$ while the tangential component is continuous. Use this to compute the surface charge distribution $\sigma(\theta)$. 1pt(s)

Hint: The first Legendre polynomials are given by

$$P_0(x) = 1, \;\; P_1(x) = x, \;\; P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{3}$$

Furthermore they obey the orthogonality relation

$$\int_{-1}^{1} dx P_l(x) P_m(x) = \frac{2}{2l+1}\delta_{lm}. \tag{4}$$

Problem 4.3: Conducting tip* [**Written** | 3 bonuspt(s)]

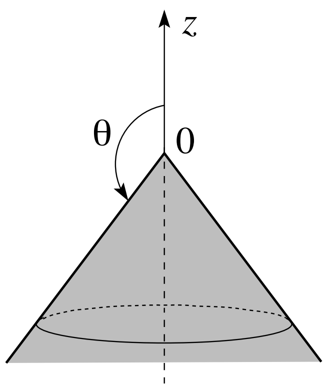
ID: ex_conducting_tip:edyn24

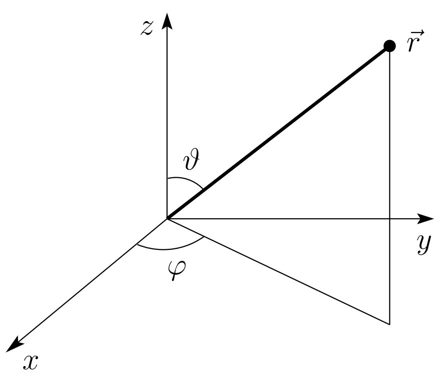
Learning objective

Spherical coordinates can also be used to solve more complex geometries like a conducting tip. The latter can be described as a perfect cone. Following the lecture notes, we derive this beautiful result. It explains, why lightning prefers to strike sharp objects, as the electric field is strongly enhances near the surface.

a) First, write the Laplace equation, $\nabla^2\Phi = 0$, in spherical coordinates and make an ansatz with separation of variables: $\Phi(\boldsymbol{r}) = u(r)P(\vartheta)\chi(\phi)/r$. Derive the differential equation for $P(\vartheta)$ and regard it as an eigenvalue problem. In the lecture notes, we take only solutions $l = 0, 1, 2, \ldots$ (Legendre polynomials $P_l(z), z \equiv \cos\vartheta$). Which solutions does one find for $l = -1, -2, \ldots$ and l non-integer ? How are they called ? When is one allowed to choose solutions with l non-integer ? 1pt(s)

Consider now the grounded conical tip (or cut) with angle $0 < \theta < \pi$. We are looking for the potential $\Phi(r, \vartheta, \phi)$ outside of the tip $(0 \leq \vartheta \leq \theta)$.





b) Starting from the Legendre differential equation $P(z)$ as derived in part (a), perform a change of variables $\xi \equiv \frac{1}{2}(1 - z)$ and derive the equation in the form of $P(\xi)$ (now the separation 1pt(s) constant l is called ν). When is one allowed to set $m = 0$? Solve the differential equation using a generalized power series ansatz

$$P(\xi) = \xi^\alpha \sum_{j=0}^{\infty} a_j \xi^j .$$

For which ν do these solutions $P_\nu(z)$ have singularities on the interval $z \in [-1, 1]$? Do these singularities belong to the (physical) domain of definition ? How does the boundary condition $\Phi(\vartheta = \theta) = 0$ restrict the possible values for ν ? How do these values depend on θ ?

c) Consider a pointed tip with $180° - \theta = 5°$. Let the potential $\Phi(r = d, \vartheta = 0, \phi = 0) = \Phi_0$ at a small distance d. How large is the electric field closer to the tip at distance $d' = \frac{d}{10}$? **1**pt(s)