Prof. Dr. Hans-Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

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Problem 10.1: Ideal waveguide, part 1

[**Oral** | 4 pt(s)]

ID: ex_ideal_waveguide_part_1:edyn24

Learning objective

The problem set deals with wave guides, i.e. structures which confine the propagation of electromagnetic waves to one direction. We start with analyzing the ideal wave guide. Here, in the first part of the problem, we calculate modes and prove that no transverse electromagnetic waves exist in this wave guide.

We consider electromagnetic waves which are confined to an ideal, cylindrical wave guide and propagate in z-direction. The term "ideal" means that the boundary surface of the wave guide is a perfect conductor. The medium inside the cylinder is assumed to be homogeneous with permeability $\mu = \mu_0 \mu_r$ and permittivity $\varepsilon = \varepsilon_0 \varepsilon_r$. For this geometry, we can separate off the propagation in z-direction:

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{E}_0(x, y) e^{i(kz - \omega t)}, \tag{1}$$

$$\boldsymbol{H}(x, y, z, t) = \boldsymbol{H}_0(x, y) e^{i(kz - \omega t)}. \tag{2}$$

The boundary conditions at the interior boundary $\partial\mathcal{V}$ of the cylinder are $\boldsymbol{E}^{\parallel} = 0$ and $\boldsymbol{H}^{\perp} = 0$.

We distinguish between different type of electromagnetic waves: We define transverse magnetic (TM) modes and transverse electric (TE) modes via

$$E_z|_{\partial\mathcal{V}} = 0\,, \quad H_z = 0 \quad \text{(TM modes)}, \tag{3a}$$

$$\partial_n H_z|_{\partial\mathcal{V}} = 0\,, \quad E_z = 0 \quad \text{(TE modes)}. \tag{3b}$$

If both $E_z = 0$ and $H_z = 0$, we have transverse electromagnetic (TEM) modes.

a) Start with Maxwell equations and show that the z-components of the fields must fulfill the eigenvalue equations **1**pt(s)

$$\left(\nabla_t^2 + \gamma_\lambda^2\right) E_z = 0\,, \tag{4a}$$

$$\left(\nabla_t^2 + \gamma_\lambda^2\right) H_z = 0\,, \tag{4b}$$

with $\gamma_\lambda^2 = \mu_r \varepsilon_r \omega^2 / c^2 - k_\lambda^2$ (the index λ labels the eigenvalues) and $\nabla_t = \boldsymbol{e}_x \partial_x + \boldsymbol{e}_y \partial_y$. Show that the boundary conditions are $E_z|_{\partial\mathcal{V}} = 0$ and $\partial_n H_z|_{\partial\mathcal{V}} = 0$.

b) Show that the solutions for the transverse field components $\boldsymbol{E}_t = E_x \boldsymbol{e}_x + E_y \boldsymbol{e}_y$ and $\boldsymbol{H}_t = H_x \boldsymbol{e}_x + H_y \boldsymbol{e}_y$ are determined by the solutions for the z-components of the fields via **1**pt(s)

$$\boldsymbol{E}_t = \frac{ik_\lambda}{\gamma_\lambda^2} \nabla_t E_z, \quad \boldsymbol{H}_t = \frac{\varepsilon_0 \varepsilon_r \omega}{k_\lambda} \boldsymbol{e}_z \times \boldsymbol{E}_t \qquad \text{(TM modes)}, \tag{5a}$$

$$\boldsymbol{H}_t = \frac{ik_\lambda}{\gamma_\lambda^2} \nabla_t H_z, \quad \boldsymbol{E}_t = -\frac{\mu_0 \mu_r \omega}{k_\lambda} \boldsymbol{e}_z \times \boldsymbol{H}_t \qquad \text{(TE modes)}. \tag{5b}$$

c) Solve the equations for the TE modes by taking into account that the wave guide is a cylinder. Show that the solutions are given by Bessel functions. 1pt(s)

d) Show that in an ideal wave guide, no TEM modes exist. 1pt(s)

Problem 10.2: Ideal waveguide, part 2

[**Written** | 4 pt(s)]

ID: ex_ideal_waveguide_part_2:edyn24

Learning objective

Here, in the second part of the problem, we calculate the energy flow in the ideal wave guide.

Introducing the critical frequency $\omega_\lambda = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \gamma_\lambda$ allows us to write $k_\lambda^2 = \frac{\mu_r \varepsilon_r}{c^2}(\omega^2 - \omega_\lambda^2)$ for the wave number that describes the propagation along the z-axis of the waveguide.

The flow of energy is given by the *complex* Poynting vector

$$\boldsymbol{S} = \frac{1}{2} \boldsymbol{E} \times \boldsymbol{H}^* \tag{6}$$

where $*$ denotes complex conjugation.

a) Employ the results from exercise Problem 10.1 a) and b) to show that the Poynting vector takes the form 1pt(s)

$$\boldsymbol{S} = \frac{\omega k_\lambda}{2\gamma_\lambda^4} \begin{cases} \varepsilon[|\nabla_t E_z|^2 \boldsymbol{e}_z + i\frac{\gamma_\lambda^2}{k_\lambda} E_z \nabla_t E_z^*] \,, & \text{(TM modes)} \\ \mu[|\nabla_t H_z|^2 \boldsymbol{e}_z - i\frac{\gamma_\lambda^2}{k_\lambda} H_z^* \nabla_t H_z] \,. & \text{(TE modes)} \end{cases} \tag{7}$$

b) Which contribution in Eq. (7) determines the energy flow in z-direction? Integrate this part over the cross section S of the waveguide for both TE and TM modes and show that the propagating power is given by 1pt(s)

$$\begin{Bmatrix} P_{\text{TM}} \\ P_{\text{TE}} \end{Bmatrix} = \frac{c}{2\sqrt{\mu_r \varepsilon_r}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} \int_S \mathrm{d}A \begin{Bmatrix} \varepsilon |E_z|^2 \\ \mu |H_z|^2 \end{Bmatrix} . \tag{8}$$

Hint: Use Green's first identity for two scalar fields Ψ and Φ

$$\int_U \mathrm{d}V \left[\Phi \nabla^2 \Psi + \nabla \Phi \cdot \nabla \Psi\right] = \oint_{\partial U} \mathrm{d}A \, \Phi \frac{\partial \Psi}{\partial n} \tag{9}$$

and the boundary conditions given in (**??**). Here, $U \subset \mathbb{R}^n$ is some n-dimensional subset, ∂U its boundary, and $\partial_n \Psi = \boldsymbol{n} \cdot \nabla \Psi$ is the normal derivative with respect to ∂U. Eq. (**??**) may be useful as well.

c) Along the same lines, calculate the energy $U_{\text{TM/TE}}$ *per unit length* of the waveguide and show that 1pt(s)

$$\begin{Bmatrix} U_{\text{TM}} \\ U_{\text{TE}} \end{Bmatrix} = \frac{1}{2} \left(\frac{\omega}{\omega_\lambda}\right)^2 \int_S \mathrm{d}A \begin{Bmatrix} \varepsilon |E_z|^2 \\ \mu |H_z|^2 \end{Bmatrix} . \tag{10}$$

Hint: The time-averaged energy u *per volume* (energy density) is given by

$$u = \frac{1}{4} \left(\varepsilon |\boldsymbol{E}|^2 + \mu |\boldsymbol{H}|^2\right) . \tag{11}$$

d) Finally, combine the results (8) and (10) to derive an expression for the velocity of the energy flux and compare your result with the group velocity $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_\lambda}$. **1**pt(s)

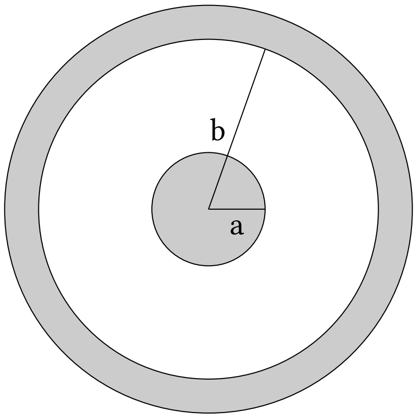
Problem 10.3: Coaxial cable

[**Oral** | 2 pt(s)]

ID: ex_coaxial_cable:edyn24

> **Learning objective**
>
> In this exercise we will have a a look at the coaxial cable as an example of a more complicated waveguide. We will see that in the medium between the wire and shielding, TEM modes can propagate.

A coaxial cable consists of an outer conducting shell with radius b and an inner conductor with radius a aligned coaxially. The space between these two components is filled with a dielectric of permeability ε and permittivity μ. We can determine the field configuration in the dielectric as we would in a waveguide.



As in exercise Problem 10.1, we consider propagating waves

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{E}_0(x, y)e^{i(kz-\omega t)}, \tag{12}$$
$$\boldsymbol{H}(x, y, z, t) = \boldsymbol{H}_0(x, y)e^{i(kz-\omega t)}. \tag{13}$$

Show that this waveguide - in contrast to the waveguide in exercise Problem 10.1 - can support a TEM mode and determine $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$ explicitly.