Information on lecture and tutorials
Here a few infos on the modalities of the course “Theo III: Elektrodynamik”:

• The COMPUS-ID of this course is 042030002.
• You can find detailed information on lecture and tutorials on the website of our institute:
  https://itp3.info/edyn24
• You can also find detailed information on lecture and tutorials on ILIAS:
  https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_3680917.html
• Written problems have to be handed in via ILIAS and will be corrected by the tutors. You must earn at least 80% of the written points to be admitted to the exam.
• Oral problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least 66% of the oral points to be admitted to the exam.
• Every student is required to present at least 2 of the oral problems at the blackboard to be admitted to the exam.
• Problems marked with an asterisk (*) are optional and can earn you bonus points.
• If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Vector Calculus
ID: ex_vector_calculus:edyn24

[Written | 4 pt(s)]

Learning objective
Recall some standard identities of vector calculus which will be used throughout the lecture.

Definitions and conventions: We write the vectorial differentiation operators grad, div, rot using the vector $\nabla$ of partial derivatives $\nabla_i := \partial / \partial x_i$ as

$$\text{grad } F := \nabla F, \quad \text{div } A := \nabla \cdot A, \quad \text{rot } A := \nabla \times A.$$  \hfill (1)

The components of a three-dimensional vector product $a \times b$ are given by

$$(a \times b)_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} a_j b_k,$$  \hfill (2)

here $\varepsilon_{ijk}$ is the totally anti-symmetric tensor in $\mathbb{R}^3$ with $\varepsilon_{123} = +1$.

a) Show that

$$\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \quad \text{and} \quad \frac{1}{2} \sum_{i,j=1}^{3} \varepsilon_{ijk} \varepsilon_{ijl} = \delta_{kl}.$$  \hfill (3)
b) Show the following identities for vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \) and \( \mathbf{d} \):

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}),
\]
\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},
\]
\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).
\]

(4a)

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
\]
\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})
\]
\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).
\]

(4b)

(4c)

c) Prove the following identities for the scalar fields \( F \) and vector fields \( \mathbf{A}, \mathbf{B} \):

\[
\nabla \times (\nabla F) = 0,
\]
\[
\nabla \cdot (\nabla \times \mathbf{A}) = 0,
\]
\[
\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A},
\]
\[
\nabla \cdot (\nabla \times \mathbf{A}) = (\nabla F) \cdot \mathbf{A} + F \nabla \cdot \mathbf{A},
\]
\[
\nabla \times (\nabla \times \mathbf{A}) = (\nabla F) \times \mathbf{A} + F \nabla \times \mathbf{A},
\]
\[
\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}),
\]
\[
\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).
\]

(5a)

(5b)

(5c)

(5d)

(5e)

(5f)

(5g)

Problem 1.2: Gauß’s Theorem

ID: ex_gauss_theorem:edyn24

Learning objective

This problem recapitulates Gauß’s theorem which will be useful for solving problems in electrostatics. We calculate the surface integrals of different vector fields over a closed surface (i.e. the flux through the surface) and show explicitly that they are equal to the volume integrals of the divergence of the fields over the region inside the surface.

Consider the following vector fields \( \mathbf{A}_i \) in two dimensions

\[
\mathbf{A}_1 = (3xy(y - x), x^2(3y - x)) \quad (6a)
\]
\[
\mathbf{A}_2 = (x^2(3y - x), 3xy(x - y)) \quad (6b)
\]
\[
\mathbf{A}_3 = (x/(x^2 + y^2), y/(x^2 + y^2)) = \mathbf{x}/|\mathbf{x}|^2 \quad (6c)
\]

a) Compute the flux of \( \mathbf{A}_i \) through the boundary of the square \( Q \) with corners \( x = (\pm 1, \pm 1) \)

\[
I_i = \oint_{\partial Q} dx \cdot \mathbf{n} \cdot \mathbf{A}_i.
\]

(7)

b) Calculate the divergence of \( \mathbf{A}_i \) and its integral over the area of this square \( Q \)

\[
I'_i = \int_Q d^2 x \nabla \cdot \mathbf{A}_i.
\]

(8)

Problem 1.3: Stokes’ Theorem

ID: ex_stokes_theorem:edyn24
Consider the vector field
\[ A = (x^2y, x^3 + 2xy^2, xyz) . \] (9)

a) Compute the integral along the circle \( S \) around the origin in the \( xy \)-plane with radius \( R \)
\[ I = \oint_S \mathbf{d}x \cdot \mathbf{A} . \] (10)

b) Calculate the curl \( \mathbf{B} \) of the vector field \( \mathbf{A} \)
\[ \mathbf{B} = \nabla \times \mathbf{A} . \] (11)

c) Determine the flux of the curl \( \mathbf{B} \) through the disk \( D \) whose boundary is \( S, \partial D = S \)
\[ I' = \int_D d^2x \mathbf{n} \cdot \mathbf{B} . \] (12)