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Problem 9.1: Elliptically polarized waves

[**Oral** | 2 pt(s)]

ID: ex_elliptically_polarized_waves:edyn23

Learning objective

In this exercise we discuss the concept of polarization for electromagnetic plane waves based on the example of elliptically polarized light.

A wave $\boldsymbol{E}(\boldsymbol{x}, t)$ with wave vector $\boldsymbol{k} = k\boldsymbol{e}_z$ is given by

$$E_x(\boldsymbol{x}, t) = A\cos(kz - \omega t) \tag{1}$$
$$E_y(\boldsymbol{x}, t) = B\cos(kz - \omega t + \phi). \tag{2}$$

a) Show that the trajectory of the vector $\boldsymbol{E}(\boldsymbol{0}, t)$, which describes the polarization of the wave, is an ellipse. For which values of A, B and ϕ is this trajectory a circle ? **1**pt(s)

Hint: Use the trigonometric addition theorem

$$\cos(\omega t - \phi) = \cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi) \tag{3}$$

in order to obtain equations of the form of a conic section,

$$ax^2 + 2bxy + cy^2 + f = 0. \tag{4}$$

Which conditions does one need to impose on a, b, c such that Eq. (4) describes an ellipse?

b) Show that for general A and B the wave can be written as the superposition of two oppositely circularly polarized waves **1**pt(s)

$$\boldsymbol{E}(\boldsymbol{x}, t) = \mathrm{Re}(\boldsymbol{E}_+(z, t) + \boldsymbol{E}_-(z, t)), \tag{5}$$

where $\boldsymbol{E}_\pm(z, t) = A_\pm \boldsymbol{\epsilon}_\pm e^{i(kz-\omega t)}$. Here $A_\pm$ are constants and $\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{e}_x \pm i\boldsymbol{e}_y)$. Determine $A_\pm$ as a function of A, B and ϕ.

Hint: Write $\boldsymbol{E}(x, t)$ as the real part of a complex vector and transform it to the form in Eq. (5).

Problem 9.2: Reflection and transmission

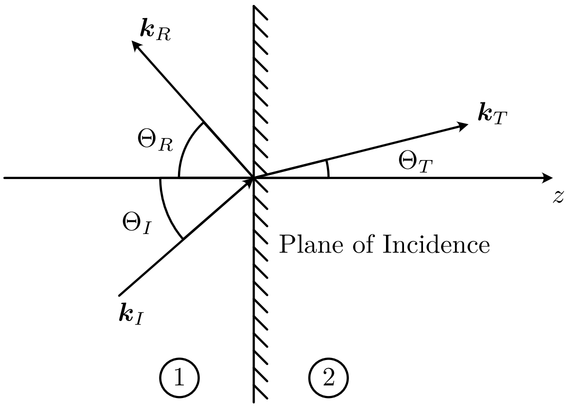
[**Written** | 4 pt(s)]

ID: ex_reflection_transmission:edyn23

Learning objective

In this exercise we study the reflection and transmission of plane waves at an interface of two non-conductive media.

The space is filled with two different non-conductive media. Consider the incident, reflected and transmitted monochromatic plane waves as in the figure below.



a) The frequency of each plane wave is fixed. Why and how are the wave numbers $\boldsymbol{k}_i$ related to each other in terms of the angles θ_i, where $i \in \{I, T, R\}$? **1** pt(s)

b) Give the boundary conditions for the electric and magnetic field at the interface and simplify them with the results from a). **1** pt(s)

In the following, we consider s-polarized light, which means that the electric field is perpendicular to the plane of incidence (also referred to as transverse electric (TE) polarization).

c) Use the boundary conditions at the interface to determine the reflection coefficient $r = E_R/E_I$ and the transmission coefficient $t = E_T/E_I$ as a function of Θ_I and the refractive indices n_i of the two media. **1** pt(s)

d) Consider a incoming plane wave with $\Theta_I = 0$ that passes from the vacuum to a medium with refractive index n' and assume $\mu_1 = \mu_2$ in the following. Find the reflected power I_R and the transmitted power I_T and show that they satisfy $I_I = I_R + I_T$. **1** pt(s)

Problem 9.3: Propagation of wave packets in media [**Oral** | 4 pt(s)]

ID: ex_propagation_wave_packets_media:edyn23

> **Learning objective**
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> In media, the propagation of wave packets exhibits a dispersion, and the propagation speed of the wave packet is no longer described by the light velocity, but rather the group velocity. In this exercise, these concepts are studied in a simple example.

We start with a general scalar field Ψ, the time evolution of which is given as a superposition of plain waves,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k \, \hat{\Psi}_0(k) \exp\left(i[kx - \omega(k)t]\right), \tag{6}$$

where $\hat{\Psi}_0(k) \equiv \mathcal{F}[\Psi_0](k)$ is the Fourier transform of the initial wave packet $\Psi_0 = \Psi(x, t = 0)$. The function $\omega = \omega(k)$ is called *dispersion relation* and determined by the differential equation that governs the dynamics of Ψ.

a) Give two paradigmatic examples of differential equations ("wave equations") with general solutions given by (6) and compare their corresponding dispersion relations $\omega = \omega(k)$. 1pt(s)

b) Assume that $\hat{\Psi}_0(k)$ is sharply peaked around k_0. Then it is reasonable to expand $\omega(k)$ at k_0 for small $k - k_0$ up to first order (Why?). Use this expansion in Eq. (6) to show that $\Psi(x,t)$ can be written in the form 1pt(s)

$$\Psi(x,t) = e^{i\phi(x - v_p t)}\psi(x - v_g t), \tag{7}$$

where $\phi(x)$ is a real function and $\psi(x)$ an arbitrary scalar field. Give expressions for v_p and v_g in terms of $\omega(k)$. v_p and v_g are called *phase*- and *group* velocity, respectively.

In the following we focus on a special case, namely a Gaussian wave packet at $t = 0$,

$$\Psi_0(x) \equiv \Psi(x, t = 0) = \psi_0 \exp\left(-\frac{x^2}{2\sigma_x^2}\right), \tag{8}$$

the propagation of which is still governed by an arbitrary dispersion relation $\omega = \omega(k)$. Here, σ_x^2 is the variance that describes the width of the wave packet.

c) Show that the initial wave packet in Fourier representation $\hat{\Psi}_0(k)$ is Gaussian as well, i.e., 1pt(s)

$$\hat{\Psi}_0(k) = \hat{\psi}_0 \exp\left(-\frac{k^2}{2\sigma_k^2}\right). \tag{9}$$

What is the relation between σ_x and σ_k and how can one interpret this result?

Hint: Use $\int_{\mathbb{R}} dx\, e^{-\frac{x^2}{2\sigma^2}} = \sqrt{2\pi}\sigma$ by completing the square.

d) Assume that $\hat{\Psi}_0(k)$ is peaked such that an expansion of $\omega(k)$ up to second order in k is a valid approximation, 1pt(s)

$$\omega(k) \approx \omega_0 + v_g k + \frac{1}{2} w_g k^2. \tag{10}$$

What is the requirement on σ_x for $\hat{\Psi}_0(k)$ to be sharply "peaked"? w_g is called *group velocity dispersion*. How does it relate to v_g?

Use Eq. (6) and your result from (c) to calculate $\Psi(x,t)$ explicitly.