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Problem 5.1: Potential of an infinitely long cylinder

[**Oral** | 2 pt(s)]

ID: ex_potential_infinite_cylinder:edyn23

> **Learning objective**
>
> Here, we solve Laplace's equation in cylindrical coordinates with Dirichlet boundary conditions.

We consider an infinitely long, hollow cylinder of radius R. Using Laplace's equation in cylindrical coordinates, we determine the electric potential inside and outside of the cylinder, given the value of the potential on the boundary of the cylinder.

a) Assume the potential on the boundary is given by **1**pt(s)

$$\phi(z, \varrho = R, \varphi) = \phi_0 + \phi_1 \cos\varphi \;, \tag{1}$$

where z is the axial coordinate, φ is the polar angle, and ϱ the radial distance in cylindrical coordinates. Think about the geometry of the problem and calculate the potential inside and outside of the cylinder.

b) The potential on the boundary of the cylinder is **1**pt(s)

$$\phi(z, \varrho = R, \varphi) = \cos(kz)\,(\phi_0 + \phi_1 \cos\varphi) \;, \tag{2}$$

with $k \neq 0$. Calculate the potential and determine its value in the limit $\varrho \to \infty$ (for taking this limit, it is helpful to look up the asymptotic behavior of the Bessel function e.g. on Wikipedia).

Problem 5.2: Electric field of a dipole

[**Written** | 6 pt(s)]

ID: ex_electric_field_dipole:edyn23

> **Learning objective**
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> In the first part of the problem, we calculate the electric field for a dipole. The resulting expression contains a δ-function term, whose physical importance is discussed in the second part of the problem.

a) Recall the important result $\nabla^2 \frac{1}{|\boldsymbol{r}|} = -4\pi\delta^3(\boldsymbol{r})$ from Problem 2.1 and generalize it to **1**pt(s)

$$\partial_\alpha \partial_\beta \frac{1}{|\boldsymbol{r}|} = -\frac{\delta_{\alpha\beta}}{|\boldsymbol{r}|^3} + 3\frac{x_\alpha x_\beta}{|\boldsymbol{r}|^5} - \frac{4\pi}{3}\delta_{\alpha\beta}\,\delta^3(\boldsymbol{r}). \tag{3}$$

Hint: Use a symmetry argument and the result from exercise Problem 2.1 to derive the last term in equation (3).

b) In the lecture, it was demonstrated that the electric potential for a dipole $\boldsymbol{p}$ is given by $\phi(\boldsymbol{r}) = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0|\boldsymbol{r}|^3} = -(\boldsymbol{p}\cdot\boldsymbol{\nabla})\frac{1}{4\pi\epsilon_0|\boldsymbol{r}|}$. Using relation (3), show that the electric field of the dipole can be written as ($\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$): **1pt(s)**

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\hat{\boldsymbol{r}}\cdot\boldsymbol{p})\,\hat{\boldsymbol{r}} - \boldsymbol{p}}{|\boldsymbol{r}|^3} - \frac{4\pi}{3}\boldsymbol{p}\,\delta^3(\boldsymbol{r})\right]. \tag{4}$$

The δ-function term in equation (4) is a correction for $\boldsymbol{r} = 0$. In the following, we are going to re-derive it in a different way to understand its physical origin.

We would like to prove the following Theorem: The *average electric field* over the volume V enclosed by a sphere of radius R, due to an arbitrary charge distribution within the sphere, is given by

$$\overline{\boldsymbol{E}} = -\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}}{R^3}, \tag{5}$$

where $\boldsymbol{p}$ is the total dipole moment with respect to the center of the sphere.

c) To do this, first calculate the average electric field within the sphere (with enclosed volume V), due to a single charge q at position $\boldsymbol{r}_q$: **1pt(s)**

$$\overline{\boldsymbol{E}}_q = \frac{1}{V}\int_V \mathrm{d}^3r\,\boldsymbol{E}_q(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{V}\int_V \mathrm{d}^3r\,\frac{\boldsymbol{r}-\boldsymbol{r}_q}{|\boldsymbol{r}-\boldsymbol{r}_q|^3}. \tag{6}$$

Realize that this expression can also be considered as the electric field *at the position* $\boldsymbol{r}_q$, that is generated by a (fictional) ball with a uniform charge density $\rho = -q/V$. Use this analogy to calculate $\overline{\boldsymbol{E}}_q$ via Gauss's law.

d) Use the superposition principle to generalize the result for the point charge q to arbitrary charge distributions and prove equation (5). **1pt(s)**

e) Explicitly calculate the average electric field that is generated by a point-like dipole, by integrating the electric field from equation (4) over a ball. In your integration, start by excluding a small region around the origin. **1pt(s)**

f) Finally, show that the δ-function term in equation (5) is essential to satisfy the average-value theorem. **1pt(s)**

Note: Another approach is to calculate the electric field of a homogeneously polarized ball of radius a. Outside of the ball, the field is exactly given by equation (4). Inside the ball, the field has a constant value $\boldsymbol{E}_{\text{in}} = -1/4\pi\epsilon_0 \cdot \boldsymbol{p}/a^3$, where $\boldsymbol{p}$ is the dipole moment of the ball. As the size of the ball goes to zero, the field strength goes to infinity in such a way that the integral over the ball remains constant, giving the prefactor of the δ-function: $-\boldsymbol{p}/3\epsilon_0$.

Problem 5.3: Spherical multipole moment

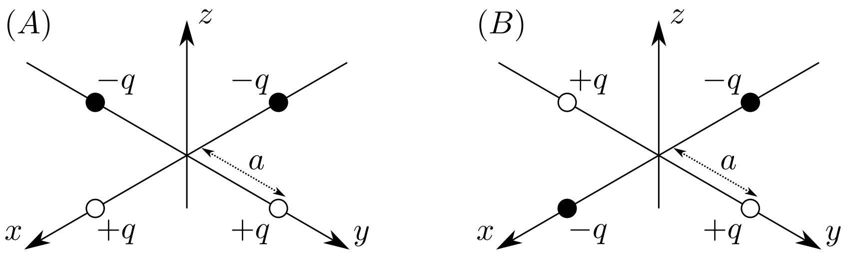
[**Oral** | 2 pt(s)]

ID: ex_spherical_multipole_moment:edyn23

Learning objective

The goal of this problem is to calculate the spherical multipole moments q_{lm} for different charge distributions and to study when a quadrupole moment occurs.

We perform calculations for two charge distributions (A) and (B). Both consist of four charges in the xy-plane, placed distance a from the origin and equidistant to each other. The distributions are given in the sketch



a) Write down the charge distribution in spherical coordinates. The relation between the charge distribution in Cartesian coordinates $\rho(x, y, z)$ and spherical coordinates $\rho_{\text{sph}}(r, \theta, \phi)$ is given by (why?): **1** pt(s)

$$\rho(x, y, z) = \frac{\rho_{\text{sph}}(r, \theta, \phi)}{r^2 \sin\theta} \tag{7}$$

b) Compute the spherical monopole, dipole and quadrupole moments for both arrangements. **1** pt(s)