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Problem 3.1: Method of images, Part 1

[**Oral** | 4 pt(s)]

ID: ex_method_images_part1:edyn23

Learning objective

We apply the method of images to solve complex boundary-value problems in electrostatics.

Consider a conducting plane (yz-plane at $x = 0$) and a conducting sphere of radius $R \ll L$ with the center at $(L, 0, 0)$. A point charge q_1 is located between the plane and the sphere at $(r, 0, 0)$; $0 < r < L - R$. Your goal is to determine the electrostatic potential using *image charges* in lowest order in R/L. In order, to satisfy boundary conditions on two conducting objects it is necessary to construct a complete set of mirror images.

Hint: Note that $L - r$ can be of the order of R.

a) First, find the image charge q_2 of q_1 which produces the correct boundary conditions on the $yz-$plane. Next, find image charges of q_1 and q_2 (call them q_3 and q_4, respectively) so that the boundary conditions on the conducting sphere are satisfied. **1**pt(s)

b) Determine the image charges of q_3 and q_4 (call them q_5 and q_6, respectively). **1**pt(s)

c) All other image charges are of higher order in R/L. Take them into account assuming that there are two sets of charges at positions $x_\pm = \pm L$. For each position, sum up a geometric series of image charges in the limit of $R \ll L$. **1**pt(s)

d) Determine the electrostatic potential. **1**pt(s)

Problem 3.2: Method of images, Part 2

[**Oral** | 3 pt(s)]

ID: ex_method_images_part2:edyn23

Learning objective

In the previous exercise, the method of images was used to calculate the potential of a point charge in the vicinity of boundary surfaces. In this exercise you will see that the method of images can be used to calculate the influence of a boundary surface on a uniform electric field as well.

Consider a conducting sphere of radius a centered in the origin in a uniform electric field E. Your goal is to determine the electrostatic potential generated by the sphere using the method of images.

The uniform field can be thought of as being produced by appropriate positive and negative charges at infinity.

a) What is the relation between the charges ($\pm Q$) and distances ($\pm R$) which generate the electric field E in the origin without the sphere? **1**pt(s)

b) Determine the electrostatic potential due to charges and their images for finite R in presence of the sphere. **1**pt(s)

c) Determine the limit $R \to \infty$ of the potential from part b). **1**pt(s)

Problem 3.3: Conformal mapping

[**Written** | 3 pt(s)]

ID: ex_conformal_mapping:edyn23

> **Learning objective**
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> Like the previous exercises, this exercise in principle can be solved using the method of images. We will use this exercise, however, to demonstrate how conformal mappings and complex potentials can be used to simplify 2-dimensional problems in electrostatics.

Imagine that we know the complex potential $w(z)$ for a 2-dimensional electrostatic problem with point charges and conducting objects. Next, we change only the geometry of the problem by a conformal map f. If f^{-1} is also conformal in the region of our interest, then the potential of the new complex problem is given by $w(f^{-1}(z))$.

Solve the following problem using conformal mapping: A circular conductor with radius $R = 1$ at position $(0, 0)$ and the point charge q at position $\boldsymbol{r}_1 = (\xi < 1, 0)$ are given.

a) Calculate the complex potential for a charge at position $\boldsymbol{r}_0 = (0, 0)$ inside the circular conductor. **1**pt(s)

b) Find a conformal mapping of the form $f(z) = \frac{z+b}{cz+d}$ which maps the unit circle onto itself and $\boldsymbol{r}_0$ onto $\boldsymbol{r}_1$. **1**pt(s)

c) Determine the complex and electrostatic potentials. **1**pt(s)

Problem 3.4: Complex potential*

[**Written** | 3 bonuspt(s)]

ID: ex_complex_potential:edyn23

> **Learning objective**
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> The goal of this exercise is to expand on the topic of complex potentials and complex analysis to tackle involved 2-dimensional problems. It is a beautiful example where the correct treatment of the brunch cut of the square root function plays a crucial role in physics. Students interested in theoretical physics should consider this exercise to be mandatory, since it requires the fundamentals of complex analysis.

An infinitesimally thin, conducting band has width $2a$ in the x-direction and is infinitely long in $z-$direction. It is located in an external, uniform $\boldsymbol{E}$-field parallel to the x-axis. Your goal will be to determine the complex potential $w(z)$ and electrostatic potential $\varphi(x, y)$. In contrary to the previous exercise where we used a conformal mapping to find the complex potential, here we will use symmetries of the system and the conformality of $w(z)$.

a) What conditions have to be satisfied by electric field components E_x, E_y on the conducting band and for $x \to \pm\infty$? How do those conditions translate to the properties of $w(z)$? **1**pt(s)

b) The complex potential $w(z)$ has to be conformal on $\mathbb{C}\backslash[-a,a]$. Show that **1**pt(s)

$$w(z) = b\sqrt{z-a}\sqrt{z+a} \tag{1}$$

satisfies the conformality condition. Find $b \in \mathbb{C}$ for which all conditions from part a) are satisfied.
Hint: Recall that the square root function can be defined using $\log z$ as

$$\mathbb{C}\backslash\{0\} \in z \mapsto z^{1/2} := e^{1/2 \log z} \tag{2}$$

where we choose the branch cut to be $z \in [0, \infty[$. The complex logarithm can be expressed using real functions r, ϕ

$$\log(r(\cos\phi + i\sin\phi)) = \log r + i\phi, \qquad r \in \mathbb{R}_+, \quad \phi \in [0, 2\pi[. \tag{3}$$

On the branch cut, the square root function $z^{1/2}$ is discontinuous. Intuitively, the jump of the argument of $z^{1/2}$ is equal to π.

c) Determine the electrostatic potential $\varphi(x, y)$ and plot its field lines and curves of constant potential. **1**pt(s)