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Problem 13.1: Lorentz Group

[**Oral** | 3 pt(s)]

ID: ex_lorentz_group_ed:edyn23

Learning objective

This exercise serves to become familiar with groups and their properties. Groups play a fundamental role in all fields of physics and can tremendously simplify otherwise very challenging problems. In this exercise we will explicitly show that Lorentz transformations form a group and further analyze the properties of this group. Since the group elements are matrices here, this is a good example to understand the abstract properties of groups in an easy way.

First, let us revise the definition of a group.

Definition: A group is a set G together with an operation $\bullet$ (called the *group law* of G) that combines any two elements a and b to form another element, denoted by $a \bullet b$. To qualify as a group, the set and the operation, $(G, \bullet)$, must satisfy four requirements known as the *group axioms*:

1) **Closure**: For all $a, b \in G$, the result of the operation $a \bullet b$ is also an element of G.
2) **Associativity**: For all $a, b, c \in G$ the following relation is satisfied $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.
3) **Identity element**: There exists an element $e \in G$, such that for every element $a \in G$, the equality $e \bullet a = a \bullet e = a$ holds. Such an element is unique, and thus called *the* identity element.
4) **Inverse element**: For each $a \in G$, there exists an element $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.

For Lorentz transformations, we define G as the set of matrices characterized by the invariance of the metric tensor g of Minkowski spacetime, i.e.

$$G := \{\Lambda \in \mathbb{R}^{4\times 4} \quad | \quad \Lambda^t g \Lambda = g\}\,, \tag{1}$$

and the group operation $\bullet$ as the multiplication of matrices.

a) Show that the Lorentz group is a group. **1**pt(s)

b) Depending on $\det(\Lambda)$ and $\operatorname{sign}(\Lambda^0{}_0)$ the Lorentz group can be divided into four components. Show that *proper orthochronous Lorentz transformations*, i.e. $\det(\Lambda) = 1$ and $\operatorname{sign}(\Lambda^0{}_0) = 1$, form a group. **1**pt(s)

c) Next, proof that each of three other components does not form a group. Give an example of the combination of two out of four components which again gives a proper group. **1**pt(s)

Problem 13.2: Relativistic formulation of Maxwell's equations [**Oral** | 2 pt(s)]

ID: ex_galilean_invariance_of_handicapped_maxwell_equations:edyn23

Learning objective

In this exercise, we derive the Maxwell equations from a covariant formulation of classical electrodynamics. You will also show that by adding a quadratic term to the Lagrangian gives the electromagnetic four-potential A^μ a mass, but breaks the gauge invariance of the Lagrangian.

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J^\mu A_\mu \,, \tag{2}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength tensor, and $J^\mu = (c\rho, \boldsymbol{j})$ is the four-current.

a) Derive the Maxwell equations from the Lagrangian in Eq. (2) using the Euler-Lagrange equations. Write the two inhomogeneous Maxwell equations in their standard form by using $F_{0i} = -F^{i0} = E_i$ and $F_{ij} = F^{ij} = -\varepsilon_{ijk} B_k$, $i = x, y, z$. What happens with the two homogeneous equations? 1pt(s)

b) In the following, we set $J^\alpha = 0$ but add an additional term to the Lagrangian 1pt(s)

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\lambda^2}{4\pi} A^\beta A_\beta \,. \tag{3}$$

Show that the Lagrangian is not gauge invariant anymore. Using the Lorentz gauge $\partial_\alpha A^\alpha = 0$ calculate the equation of motion for A^α and show that $A^\alpha = \epsilon^\alpha e^{\pm i(\boldsymbol{p}\boldsymbol{x} - \omega t)}$ is a solution. Compare the dispersion relation to the relativistic energy-momentum relation. (set $\hbar = c = 1$)

Note: In this exercise you see that a gauge invariant theory requires the gauge bosons (here: photons) to be massless. While this is not a problem for the photons as they are massless, other gauge bosons like the W or Z bosons, which mediate the weak interaction in the standard model, have a mass. To restore the gauge invariance in these theories requires a trick: the famous Higgs mechanism.

Problem 13.3: Tractor beam [**Oral** | 4 pt(s)]

ID: ex_tractor_beam:edyn23

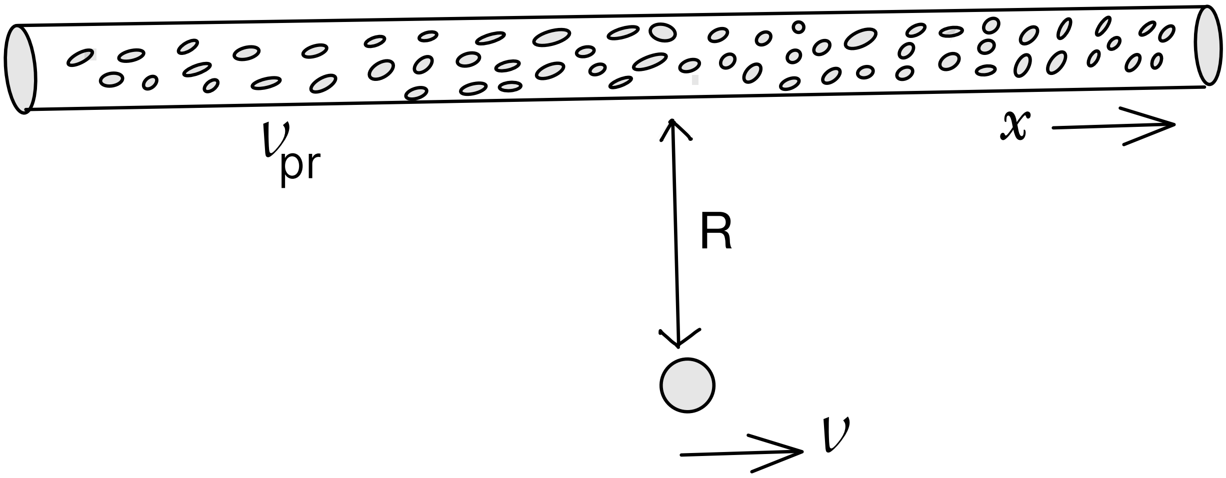
Learning objective

In this exercise, you will learn the transformation of electric and magnetic fields between different inertial frames of reference using Lorentz transformations. Later, you will calculate the equation of motion with respect to different inertial frames.

A beam of protons flies along the x-axis, wherein the protons have a speed v_{pr} and a density n (in protons per unit length).

a) Calculate the electric $\boldsymbol{E}(r)$ and magnetic fields $\boldsymbol{B}(r)$ generated by the beam as a function of the distance to the beam r. 1pt(s)
Tip: The charge distribution can be approximated for $r \gg 1/n$ as continuous, which leads to the charge density $\lambda = en$ and the current is given by $I = env_{pr}$.

b) Consider a point charge P moving parallel to the proton beam at the distance R with a velocity v. Write down the equation of motion for this moving charge P considering charge to be negative $-Q$. **1**pt(s)



c) Now we look at the same situation from the perspective of the moving charge particle P. We set here $v = v_{pr}$. Transform the electric and magnetic fields into the coordinate system K' of the moving charge particle P. **1**pt(s)

d) Derive the equation of motion in the rest frame for the moving charge P. **1**pt(s)