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Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Theo III: Elektrodynamik"***:

- The C@MPUS-ID of this course is 042900000.
- You can find detailed information on lecture and tutorials on the website of our institute:

  https://itp3.info/edyn
- You can also find detailed information on lecture and tutorials on ILIAS:

  https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_3217707.html
- **Written** problems have to be handed in and will be corrected by the tutors. You must earn at least **80 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **2** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Vector Calculus

[**Written** | 4 pt(s)]

ID: ex_vector_calculus:edyn23

Learning objective

Recall some standard identities of vector calculus which will be used throughout the lecture.

Definitions and conventions: We write the vectorial differentiation operators grad, div, rot using the vector $\boldsymbol{\nabla}$ of partial derivatives $\nabla_i := \partial/\partial x_i$ as

$$\operatorname{grad} F := \boldsymbol{\nabla} F\,, \qquad \operatorname{div} \boldsymbol{A} := \boldsymbol{\nabla}\cdot\boldsymbol{A}\,, \qquad \operatorname{rot} \boldsymbol{A} := \boldsymbol{\nabla}\times\boldsymbol{A}\,. \tag{1}$$

The components of a three-dimensional vector product $\boldsymbol{a}\times\boldsymbol{b}$ are given by

$$(\boldsymbol{a}\times\boldsymbol{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\, a_j b_k\,, \tag{2}$$

here ε_{ijk} is the totally anti-symmetric tensor in $\mathbb{R}^3$ with $\varepsilon_{123} = +1$.

a) Show that **1**pt(s)

$$\sum_{i=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \text{and} \qquad \frac{1}{2}\sum_{i,j=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ijl} = \delta_{kl}\,. \tag{3}$$

b) Show the following identities for vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$: **1** pt(s)

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})\,, \tag{4a}$$
$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\,, \tag{4b}$$
$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})\,. \tag{4c}$$

c) Prove the following identities for the scalar fields F and vector fields $\boldsymbol{A}$, $\boldsymbol{B}$: **2** pt(s)

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}F) = 0\,, \tag{5a}$$
$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) = 0\,, \tag{5b}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \Delta\boldsymbol{A}\,, \tag{5c}$$
$$\boldsymbol{\nabla}\cdot(F\boldsymbol{A}) = (\boldsymbol{\nabla}F)\cdot\boldsymbol{A} + F\boldsymbol{\nabla}\cdot\boldsymbol{A}\,, \tag{5d}$$
$$\boldsymbol{\nabla}\times(F\boldsymbol{A}) = (\boldsymbol{\nabla}F)\times\boldsymbol{A} + F\boldsymbol{\nabla}\times\boldsymbol{A}\,, \tag{5e}$$
$$\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\,, \tag{5f}$$
$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})\,. \tag{5g}$$

Problem 1.2: Gauß's Theorem

[**Oral** | 2 pt(s)]

ID: ex_gauss_theorem:edyn23

Learning objective

This problem recapitulates Gauß's theorem which will be useful for solving problems in electrostatics. We calculate the surface integrals of different vector fields over a closed surface (i.e. the flux through the surface) and show explicitely that they are equal to the volume integrals of the divergence of the fields over the region inside the surface.

Consider the following vector fields $\boldsymbol{A}_i$ in two dimensions

$$\boldsymbol{A}_1 = \big(3xy(y-x), x^2(3y-x)\big)\ , \tag{6a}$$
$$\boldsymbol{A}_2 = \big(x^2(3y-x), 3xy(x-y)\big)\ , \tag{6b}$$
$$\boldsymbol{A}_3 = \big(x/(x^2+y^2), y/(x^2+y^2)\big) = \boldsymbol{x}/|\boldsymbol{x}|^2\,. \tag{6c}$$

a) Compute the flux of $\boldsymbol{A}_i$ through the boundary of the square Q with corners $\boldsymbol{x} = (\pm 1, \pm' 1)$ **1** pt(s)

$$I_i = \oint_{\partial Q} dx\, \boldsymbol{n}\cdot\boldsymbol{A}_i\,. \tag{7}$$

b) Calculate the divergence of $\boldsymbol{A}_i$ and its integral over the area of this square Q **1** pt(s)

$$I_i' = \int_Q d^2x\, \boldsymbol{\nabla}\cdot\boldsymbol{A}_i\,. \tag{8}$$

Problem 1.3: Stokes' Theorem

[**Oral** | 3 pt(s)]

ID: ex_stokes_theorem:edyn23

Learning objective

This problem recapitulates Stokes' theorem which has important applications in electromagnetism. We calculate the line integral of a vector field around the boundary of a surface and show explicitely that it is equal to the integral of the curl of the field over the surface.

Consider the vector field

$$\boldsymbol{A} = \left(x^2 y, x^3 + 2xy^2, xyz\right) \; . \tag{9}$$

a) Compute the integral along the circle S around the origin in the xy-plane with radius R **1**pt(s)

$$I = \oint_S d\boldsymbol{x} \cdot \boldsymbol{A} \, . \tag{10}$$

b) Calculate the curl $\boldsymbol{B}$ of the vector field $\boldsymbol{A}$ **1**pt(s)

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \, . \tag{11}$$

c) Determine the flux of the curl $\boldsymbol{B}$ through the disk D whose boundary is S, $\partial D = S$ **1**pt(s)

$$I' = \int_D d^2x \, \boldsymbol{n} \cdot \boldsymbol{B} \, . \tag{12}$$