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Problem 6.1: Clebsch-Gordan coefficients and spin-orbit coupling [**Oral** | 6 pt(s)]

ID: ex_clebsch_gordan_coefficients_spin_orbit_coupling:aqt2526

> **Learning objective**
>
> In this problem you will apply the angular momentum addition theorem. As an important use case, we consider the spin-orbit coupling in the hydrogen atom, which is the leading relativistic correction.

The spin-orbit coupling between the electron's spin $\boldsymbol{S}$ and the orbital angular momentum $\mathbf{L}$ for a hydrogen atom is given by the Hamiltonian

$$H_{\text{LS}} = f(r)\,\boldsymbol{L}\cdot\boldsymbol{S} = f(r)\sum_{\alpha=x,y,z} L_\alpha \otimes S_\alpha\,, \tag{1}$$

where $f(r) = e^2/2m_e^2c^2r^3$. The spin-orbit coupling can be seen as a perturbation to the non-relativistic Hamiltonian $H_0 = \boldsymbol{P}^2/2m - e^2/r$ of the hydrogen atom.

a) Define the total angular momentum operator as **2**pt(s)

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{L}\otimes\mathbb{1} + \mathbb{1}\otimes\boldsymbol{S} \tag{2}$$

and show that $\boldsymbol{J}^2$ and J_z commute both with H_0 and H_{LS}.

b) Consider the subspace with orbital angular momentum ℓ and spin s. We can write the eigenstates $|j,m\rangle$ of $\boldsymbol{J}^2$ and J_z as linear combinations of L_z- and S_z-eigenstates $|m_\ell, m_s\rangle = |\ell, m_\ell\rangle \otimes |s, m_s\rangle$, **2**pt(s)

$$|j,m\rangle = \sum_{m_\ell, m_s} c(m_\ell, m_s; j, m)\,|m_\ell, m_s\rangle\,. \tag{3}$$

The coefficients c are called *Clebsch-Gordan coefficients.* Due to their ubiquity in quantum physics there are comprehensive tables available, e.g.,

`http://pdg.lbl.gov/2011/reviews/rpp2011-rev-clebsch-gordan-coefs.pdf`.

Use this table to write down the change of basis (3) in the subspace with $\ell = 1$ and $s = 1/2$ explicitely.

c) Derive the Clebsch-Gordan coefficients in b) by hand. **2**pt(s)

Hint: Start with the *stretched state* $|j = 3/2, m_j = 3/2\rangle$ and use the ladder operator $J_- = J_x - iJ_y$ which acts as

$$J_-\,|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j, m-1\rangle\,. \tag{4}$$

Problem 6.2: System of three interacting spin-1/2 particles [**Oral** | 4 pt(s)]

ID: ex_system_three_interacting_spin_one_half_particles:aqt2526

Learning objective

In this exercise you will be going through an instructive example of a few particle spin system. The task is to show that using the theorem of addition of angular momentum, it is possible to analytically determine the ground state of three spins coupled antiferromagnetically.

Let us consider a system composed of three spin-1/2 particles.

a) What is the dimension of the Hilbert space? $2^{\text{pt(s)}}$

The total spin operator can be defined as $\mathbf{S} = \sum_{i=1}^{3} \mathbf{S}^{(i)}$ and its z projection as $S_z = \sum_{i=1}^{3} S_z^{(i)}$. What are the eigenvalues and eigenstates of $\mathbf{S}^2$ and S_z? Express the states in the spin-1/2 basis $\left|S_z^{(1)}, S_z^{(2)}, S_z^{(3)}\right\rangle$.

b) The Hamiltonian of the system is $2^{\text{pt(s)}}$

$$H = J \sum_{i=1}^{3} \mathbf{S}^{(i)} \cdot \mathbf{S}^{(i+1)}, \qquad J > 0.$$

Here we assume a periodic system (for $i = 3$ take $i + 1 = 1$). Calculate the eigenstates and eigenenergies of this Hamiltonian.

Hint: Rewrite H as a function of S^2 and $S^{(i)2}$.

Problem 6.3: Time-reversal symmetry

[Written | 10 pt(s)**]**

ID: ex_time-reversal_symmetry:aqt2526

Learning objective

According to Wigner's theorem, symmetry operators fall into two categories: unitary and anti-unitary operators. While symmetries are mostly described by unitary operators (e.g. $U(1)$ symmetry, rotational symmetry, translation symmetry), time-reversal symmetry is a fundamental (discrete) symmetry that is represented by an anti-unitary operator. In this problem, you will study the physics of time reversal for simple examples.

In the following, we consider a system with a time-independent Hamiltonian H that is invariant under time reversal given by the operator T. Since T is connected to a symmetry of the system, it commutes with the Hamiltonian, $[H, T] = 0$. The transformation of the time evolution operator $U(t)$ under time reversal is given by

$$T^{-1} U(t) T = U(-t)\,. \tag{5}$$

a) Show by using $U(t) = \exp(-\frac{i}{\hbar} H t)$ that T is an anti-linear operator. Since T is anti-linear, Wigner's theorem implies that T is an anti-unitary operator. $2^{\text{pt(s)}}$

Show further that if $|\psi\rangle$ is a solution of the Schrödinger equation, $T|\psi\rangle$ is a solution of the Schrödinger equation with $t \to -t$. Thus, $T|\psi\rangle$ satisfies the equation $-i\hbar \partial_t T|\psi\rangle = HT|\psi\rangle$.

Hint: An anti-linear operator has the property that $T(c|v\rangle) = c^* T|v\rangle$ for $c \in \mathbb{C}$ and $|v\rangle \in \mathcal{H}$ with some Hilbert space $\mathcal{H}$.

b) For spinless particles, the time-reversal operator T in the position basis satisfies **2**pt(s)

$$T\,|x\rangle = |x\rangle\ . \tag{6}$$

Show that $T\psi(x) = \psi^*(x)$. In order to do so, consider the action of T on some arbitrary state $|\psi\rangle$ and use $\psi(x) = \langle x|\psi\rangle$. It thus follows that in the position representation for spinless particles, $T = K$, where K denotes the complex conjugation with $Kc = c^*K$ for $c \in \mathbb{C}$. Show that consequently for spinless particles $T^2 = 1$.

c) Derive the transformation laws for the position, momentum and angular momentum operator in the position representation. How does this connect to classical physics? Show that the system is time-reversal invariant if the Hamiltonian is real, i.e. $H^* = H$. **2**pt(s)

d) Consider now a spin-1/2 particle. In this case, the time-reversal operator can be written as **2**pt(s)

$$T = \exp\left(-i\frac{\pi}{2}\sigma_y\right)K = -i\sigma_y\,K\,, \tag{7}$$

where σ_y is a Pauli matrix. Derive the transformation of the spin $\mathbf{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)^T$ under the transformation (7) and show that $T^2 = -I$, where I is the identity operator.

e) Show that in a system that is time-reversal invariant and $T^2 = -I$ (as for example for a spin-1/2 particle), all energy levels are (at least) doubly degenerate. This is known as *Kramers' theorem.* **2**pt(s)