Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

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Problem 5.1: Bound states of a spherical potential well

[**Written** | 6 pt(s)]

ID: ex_bound_states_spherical_potential_well:aqt2526

Learning objective

Here we will derive the bound states of a spherically symmetric potential well. To do so, we will exploit the rotation symmetry of the problem and show that the radial solutions are given by *spherical Bessel functions* (see lecture). The idea is to explicitly derive the transcendental equation that determines the eigenenergies for bound states with angular momentum $l = 0$.

Consider the Hamiltonian of a particle in three dimensions

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}) \tag{1}$$

with the spherically symmetric and piecewise constant potential

$$V(\boldsymbol{r}) = V(r) = \begin{cases} 0 & r > R \\ V_0 & r \leq R \end{cases} \tag{2}$$

with $r = |\boldsymbol{r}|$, $R > 0$ the radius of the potential well and $V_0 < 0$ the potential depth.

Your goal is to find the bound states and eigenenergies of this system and the conditions that are necessary for their existence.

a) Make the separation ansatz $\Psi(\boldsymbol{r}) = R_l(r) \cdot Y_{lm}(\theta, \varphi)$ with spherical harmonics Y_{lm} and show that the eigenvalue problem reduces to **2**pt(s)

$$\left[\rho^2 \partial_\rho^2 + 2\rho\partial_\rho + \rho^2 - l(l+1)\right] \tilde{R}_l(\rho) = 0 \tag{3}$$

with $\rho \equiv K_r r$ and $\tilde{R}_l(\rho) \equiv R_l(r)$ where $K_r \equiv \sqrt{\frac{2m(E-V(r))}{\hbar^2}}$.

b) Write down the general solution of the radial problem in the two regions $r > R$ and $r \leq R$ for a given angular momentum l and formulate the continuity and boundary conditions that the eigenstates must satisfy. **2**pt(s)

Hint: Use that the solutions of the differential equation

$$\left[x^2 \partial_x^2 + 2x\partial_x + x^2 - l(l+1)\right] y(x) = 0 \tag{4}$$

are given by the *spherical Bessel functions*

$$j_l(x) = (-x)^l \left(\frac{1}{x}\partial_x\right)^l \frac{\sin(x)}{x} \quad \text{and} \quad y_l(x) = -(-x)^l \left(\frac{1}{x}\partial_x\right)^l \frac{\cos(x)}{x} \tag{5}$$

for $l \in \mathbb{N}_0$. (The functions y_l are sometimes denoted n_l and referred to as *spherical Neumann functions*.) Write the eigenstates in terms of these functions.

c) Consider the simplest case for $l = 0$. Find explicit expressions for the bound states and derive a transcendental equation to determine their eigenenergies. At which potential depth V_0 appears the first bound state? $2^{\mathrm{pt(s)}}$

Hint: Use your knowledge of the one-dimensional potential well to analyze the transcendental equation.

Problem 5.2: Angular momentum commutation relations [**Oral** | 4 pt(s)]

`ID: ex_j_angularmo_relations:aqt2526`

Learning objective

In this exercise, you will prove the commutation relations that were stated in the lecture.

A generalized angular momentum $\boldsymbol{J} = (J_x, J_y, J_z)$ operator has the following property

$$[J_k, J_l] = i\hbar\epsilon_{klm}J_m \quad k, l, m = x, y, z. \tag{6}$$

If $\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x \pm iJ_y = J_\mp^\dagger$, show the following:

a) $[J_z, J_\pm] = \pm\hbar J_\pm$ $1^{\mathrm{pt(s)}}$

b) $[J_+, J_-] = 2\hbar J_z$ $1^{\mathrm{pt(s)}}$

c) $J_\pm J_\mp = \boldsymbol{J}^2 - J_z(J_z \mp \hbar)$ $1^{\mathrm{pt(s)}}$

d) $[\boldsymbol{J}^2, J_\pm] = 0$ $1^{\mathrm{pt(s)}}$

Problem 5.3: Rashba spin-orbit coupling and Zitterbewegung [**Oral** | 4 (+3 bonus) pt(s)]

`ID: ex_rashba_soc:aqt2526`

Learning objective

Coupling between spin and orbital degrees of freedom leads to a plethora of interesting phenomena in quantum mechanics and condensed matter theory. In this exercise, we explore it by studying a toy model that incorporates Rashba spin-orbit coupling.

We consider a two-dimensional model described by the effective Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m}\sigma_0 + \alpha[\boldsymbol{\sigma}\times\boldsymbol{p}]_z = \frac{\boldsymbol{p}^2}{2m}\sigma_0 + \alpha(\sigma_x p_y - \sigma_y p_x) = \begin{pmatrix} (p_x^2 + p_y^2)/(2m) & \alpha(p_y + ip_x) \\ \alpha(p_y - ip_x) & (p_x^2 + p_y^2)/(2m) \end{pmatrix}, \tag{7}$$

where the spin-orbit coupling is given by the Rashba Hamiltonian $\alpha[\boldsymbol{\sigma}\times\boldsymbol{p}]_z$.

a) Find the eigenvalues and eigenstates of the Hamiltonian. $2^{\mathrm{pt(s)}}$

b) In the Heisenberg picture, derive the equations of motion for operators $\boldsymbol{r}(t)$, $\boldsymbol{p}(t)$, $\boldsymbol{\sigma}(t)$. $2^{\mathrm{pt(s)}}$

*c) For the spin-polarized Gaussian wavepacket, $+3^{\mathrm{pt(s)}}$

$$\Psi(\boldsymbol{r}) = \frac{1}{\sqrt{\pi}d}e^{-\boldsymbol{r}^2/(2d^2)}e^{ik_0x}\begin{pmatrix}1\\0\end{pmatrix}, \tag{8}$$

calculate the expectation value of $\langle \boldsymbol{r}(t) \rangle$ in the limit of large $dk_0 \gg 1$. You may use either the Schrödinger or Heisenberg picture.

Hint: You may find it useful to express the wavepacket as a Fourier transform,

$$\Psi(\boldsymbol{r}) = \frac{d}{2\pi\sqrt{\pi}} \int d^2k e^{i\boldsymbol{k}\boldsymbol{r}} e^{-(k_x-k_0)^2 d^2/2} e^{-k_y^2 d^2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{9}$$