

**Problem 3.1: Translation Operator**

[Oral | 10 pt(s)]

ID: ex\_translation\_qm:aqt2526

**Learning objective**

The main idea behind this exercise is to get comfortable with the translation operator. As we will see in the next problem, this will be quite useful. Here, we take a complementary approach to how these operators are presented in the lectures.

Let  $\hat{T}$  be the translation operator. It may be defined by

$$\hat{T}(a)|x\rangle = |x+a\rangle \tag{1}$$

where  $|x\rangle$  is an eigenstate of the position operator and  $a \in \mathbb{R}$ .

- a) Show that  $\hat{T}(a)$  has the following effect on the wave function: 2<sup>pt(s)</sup>

$$\hat{T}(a)\psi(x) = \psi(x-a). \tag{2}$$

- b) Show that the translation operator commutes with the momentum operator. In order to accomplish this you should introduce the Taylor expansion of  $\psi(x-a)$  near  $x$ . 2<sup>pt(s)</sup>
- c) Show that  $\hat{T}(a)$  is unitary, i.e.,  $\hat{T}(a)^\dagger \hat{T}(a) = \mathbb{I}$ . Next, consider the eigenvalue equations for the translation operator, 2<sup>pt(s)</sup>

$$\hat{T}(a)\Phi(x) = \lambda_a\Phi(x) = \Phi(x-a) \tag{3}$$

where  $\lambda_a$  and  $\Phi_x$  are the eigenvalue and the eigenfunction of the translation operator, respectively. Find  $\lambda_a$  and  $\Phi(x)$ .

Hint: Use the Taylor expansion of  $T(a)$  to determine  $\lambda_a$ .

- d) Proof the subsequent identity for the position operator  $\hat{x}$ : 2<sup>pt(s)</sup>

$$\hat{T}(a)\hat{x}\hat{T}(a)^\dagger = \hat{x} - a\mathbb{I} \tag{4}$$

- e) Calculate the commutator  $[\hat{x}, \hat{T}(a)]$ . 2<sup>pt(s)</sup>

**Problem 3.2: Application of the translation operator**

[Oral | 6 pt(s)]

ID: ex\_ionic\_potential\_id\_translation:aqt2526

**Learning objective**

In this exercise you will investigate an application of the use of translation operator in the context of solid state physics, where regularly spaced atoms are under the influence of a potential giving rise to band-gaps.

Atoms are arranged in one dimension, with spacing  $a$ . The ionic potential is modeled by  $\delta$ - functions and strength  $V_0 > 0$ , giving the one-particle Hamiltonian

$$H = T + V(x) = \frac{p^2}{2m} + aV_0 \sum_{n \in \mathbb{Z}} \delta(x - na). \tag{5}$$

- a) Solve the equation above in the interval  $0 < x < a$ . Because of the periodicity of the atomic potential (and consequently of the Hamiltonian),  $[H, T(a)] = 0$ , the operators  $T(a)$  and  $H$  share a common eigenbasis. Use this property and the eigenequation of the translation operator,  $\Psi(x') = \Psi(x + na) = T(-na)\Psi(x) = e^{iqna}\Psi(x)$ , to show that candidate eigenfunctions of  $H$  are of the form

$$\Psi(x'(x)) = e^{iqna} (A_0 e^{ikx} + B_0 e^{-ikx}), \quad 0 < x < a, \tag{6}$$

where  $q \in \mathbb{R}$  parametrizes the eigenvalues of the translation operator. What is the relation between  $k$  and  $E$ ?

Note: “candidate wave functions” in this context means that we can use this wave function at an arbitrary  $x' = x + na$ , where  $n \in \mathbb{Z}$ , i.e. that we have a wave function for the whole system. In this case we have constructed one with the help of the translation operator based on the solutions that we have calculated for the interval  $0 < x < a$ .

- b) So far we have ignored the locations of the  $\delta$  functions. Now we deal with these missing positions by constraining the ansatz Eq. 6 appropriately: At position  $x = na$ , formulate (dis-) continuity constraints on the (derivative) wave function. Use these requirements to build a set of equations for  $A_0$  and  $B_0$ , and show the solution of these equations can be formulated as:

$$\cos qa = \alpha \frac{\sin \beta}{\beta} + \cos \beta, \tag{7}$$

where  $\alpha = ma^2V_0/\hbar^2$  and  $\beta = \sqrt{2mEa^2/\hbar^2}$ .

- c) Because of the periodicity,  $q \rightarrow q + 2m\pi/a$ ,  $m \in \mathbb{Z}$  of the left-hand side of the obtained equation, we can choose  $q \in [-\pi/a, \pi/a]$ . For each  $q$ , there is a discrete set of solutions of the equation,  $E_1(q) \leq E_2(q) \leq E_3(q) \leq \dots$ . Plot several of the lowest solutions with respect to  $q$ .

**Problem 3.3: Coherent states**

[Written | 8 (+2 bonus) pt(s)]

ID: ex\_coherent\_states:aqt2526

**Learning objective**

The eigenstates of the harmonic oscillator Hamiltonian  $|n\rangle$ , are not eigenstates of the ladder operators. The coherent state which is an eigenstate of the annihilation operator is a useful object for example in quantum optics. In this exercise you investigate some properties of coherent states and realize their convenience especially when describing the dynamic behavior of a quantum harmonic oscillator.

For every  $\alpha \in \mathbb{C}$ , we define the coherent state

$$|\psi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n \geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{8}$$

being  $|n\rangle$  an eigenstate of the 1-dimensional harmonic oscillator Hamiltonian.

a) Verify that the coherent state  $|\psi_\alpha\rangle$  is an eigenstate of the annihilation operator where  $\alpha$  is the eigenvalue, then show that the creation operator has no eigenstates. 2pt(s)

b) Show that 2pt(s)

$$|\psi_\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle \tag{9}$$

where  $a^\dagger$  and  $a$  are creation and annihilation operators, respectively.

(Tip:  $e^{-\alpha a} |0\rangle = |0\rangle$ )

\*c) Calculate  $\langle x \rangle_\alpha = \langle \psi_\alpha | x | \psi_\alpha \rangle$ ,  $\langle p \rangle_\alpha$ ,  $\Delta x_\alpha$ ,  $\Delta p_\alpha$  and show that, for all  $\alpha \in \mathbb{C}$ , +2pt(s)

$$\Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2} \tag{10}$$

is valid, i.e., coherent states minimize the position and momentum uncertainty relation. (Tip:  $a |\psi_\alpha\rangle = \alpha |\psi_\alpha\rangle$ )

d) Show that coherent states are not orthogonal and the relation 2pt(s)

$$\langle \psi_\alpha | \psi_\beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^* \beta)} \tag{11}$$

e) Derive the time evolution of coherent state  $|\psi_\alpha(t)\rangle$  and of the expectation values  $\langle x \rangle_t$  and  $\langle p \rangle_t$  2pt(s)  
under the Hamiltonian  $H = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right)$ .