

Problem 11.1: Properties of bosonic operators

[Oral | 10 pt(s)]

ID: ex_properties_of_bosonic_operators:

Learning objective

The first part of this problem reviews properties of bosonic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

In order to solve the harmonic oscillator, one can introduce bosonic creation and annihilation operators satisfying the commutation relation

$$[b, b^\dagger] = 1. \tag{1}$$

The occupation number operator is given by $\hat{n} = b^\dagger b$ with eigenstates $|n\rangle$ and eigenvalues n .

a) Using (1), show that $b|n\rangle$ and $b^\dagger|n\rangle$ are eigenstates of \hat{n} . 2pt(s)

b) Prove the following relations: 2pt(s)

$$b|n\rangle = \sqrt{n}|n-1\rangle, \tag{2}$$

$$b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{3}$$

c) Show that there has to be a state $|G\rangle$ with $b|G\rangle = 0$ and prove that n is an integer. 2pt(s)

Hint: Use the fact that there are no states with negative norm.

d) Using the commutation relations of operators in bosonic Fock space, 2pt(s)

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0 \tag{4}$$

and the states

$$|n_1, \dots, n_i, \dots\rangle = \dots \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} \dots \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}} |0\rangle, \tag{5}$$

prove the following relations:

$$b_i |n_1, \dots, n_i, \dots\rangle = \sqrt{n_i} |n_1, \dots, n_i - 1, \dots\rangle, \tag{6}$$

$$b_i^\dagger |n_1, \dots, n_i, \dots\rangle = \sqrt{n_i + 1} |n_1, \dots, n_i + 1, \dots\rangle. \tag{7}$$

e) Show that the states (5) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum b_i^\dagger b_i$. Now take a Hilbert-space of m -modes, i.e., the basis states of the Fock space are

$$|n_1, \dots, n_m\rangle.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number N $|\Psi\rangle = \hat{n}|\Psi\rangle$?

Problem 11.2: Properties of fermionic operators

[Written | 10 pt(s)]

ID: ex_properties_of_fermionic_operators:

Learning objective

This problem reviews properties of fermionic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

Now we introduce fermionic operators, which fulfill the anti-commutation relation

$$\{a, a^\dagger\} = 1. \tag{8}$$

The occupation number operator is given by $\hat{n} = a^\dagger a$ with eigenstates $|n\rangle$ and eigenvalues n .

a) Using (8), show that $a|n\rangle$ and $a^\dagger|n\rangle$ are eigenstates of \hat{n} . 2pt(s)

b) Prove the following relations: 2pt(s)

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad a^\dagger|n\rangle = \sqrt{1-n}|n+1\rangle. \tag{9}$$

c) Show that there has to be a state $|G\rangle$ with $a|G\rangle = 0$ and a state $|H\rangle$ with $a^\dagger|H\rangle = 0$. Further show that these are the only states in the Hilbert space. Assume that n is an integer. 2pt(s)

Hint: Use the fact that there are no states with negative norm.

d) Using the anti-commutation relations of operators in fermionic Fock space, 2pt(s)

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0 \tag{10}$$

and the states

$$|n_1, \dots, n_i, \dots\rangle = \dots (a_i^\dagger)^{n_i} \dots (a_1^\dagger)^{n_1} |0\rangle, \tag{11}$$

prove the following relations:

$$a_i |n_1, \dots, n_i, \dots\rangle = \delta_{n_i,1} (-1)^{S_i} |n_1, \dots, n_i - 1, \dots\rangle, \tag{12}$$

$$a_i^\dagger |n_1, \dots, n_i, \dots\rangle = \delta_{n_i,0} (-1)^{S_i} |n_1, \dots, n_i + 1, \dots\rangle, \tag{13}$$

where $S_i = n_\infty + \dots + n_{i+1}$

e) Show that the states (11) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum a_i^\dagger a_i$. Now take a Hilbert-space of m -modes, i.e., the basis states of the Fock space are 2pt(s)

$$|n_1, \dots, n_m\rangle.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number N $|\Psi\rangle = \hat{n}|\Psi\rangle$?

Problem 11.3: Expectation values of bosonic and fermionic operators

[Oral | 6 pt(s)]

ID: ex_expectation_values_fock_space:

Learning objective

Here you will take the basis states of the bosonic and fermionic Fock space and practice the calculation of expectation values of many-particle systems.

In the following, we will work with the operators

$$\begin{aligned} A &= c_i^\dagger c_i & B &= c_i^\dagger c_i c_j^\dagger c_j \\ C &= c_i^\dagger c_j^\dagger c_j c_i & D &= c_i^\dagger c_j^\dagger c_i c_j. \end{aligned}$$

- a) Show that these operators are self-adjoint. Consider the cases $c_i = b_i$ (bosons) and $c_i = a_i$ (fermions). 2^{pt(s)}
- b) Calculate the expectation value of the operators A , B , C and D , taking the states (5), with $c_i = b_i$, and (11), with $c_i = a_i$. 2^{pt(s)}
- c) Finally, determine the matrix element 2^{pt(s)}

$$\langle m_1, \dots, m_i, \dots | c_i^\dagger c_j + c_j^\dagger c_i | n_1, \dots, n_i, \dots \rangle,$$

again both for the bosonic and fermionic Fock space and operator algebra.