Prof. Dr. Hans-Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

December 12th, 2024
WS 2024/25

Problem 9.1: The pair correlation function of a Fermi Sea [**Written** | 3 pt(s)]

ID: ex_fermi_sea_correlation:aqt2425

> **Learning objective**
>
> In this problem, you apply your knowledge of calculating expectation values of fermionic operators in second quantization to determine the pair correlation function of non-interacting fermions in free space.

Consider a gas of N identical non-interacting fermions with spin $1/2$ in free space. The pair correlation function $g_{\sigma\sigma'}(\boldsymbol{r}-\boldsymbol{r}')$ is defined as

$$g_{\sigma\sigma'}(\mathbf{r}-\mathbf{r}') = \left(\frac{2}{n}\right)^2 \langle\Phi_0|\,\Psi^\dagger_\sigma(\mathbf{r})\Psi^\dagger_{\sigma'}(\mathbf{r}')\Psi_{\sigma'}(\mathbf{r}')\Psi_\sigma(\mathbf{r})\,|\Phi_0\rangle\,, \tag{1}$$

where $|\Phi_0\rangle$ is the Fermi sea with a total density $n = n_\uparrow + n_\downarrow$ and $n_\uparrow = n_\downarrow$. The pair correlation function describes the conditional probability of finding an electron at the position $\mathbf{r}'$ in the spin state σ', when we know that the second electron is at the position $\mathbf{r}$ in the spin state σ.

a) Express the field operators in the natural basis, that is, **1**pt(s)

$$\Psi_\sigma(\boldsymbol{r}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{r}} c_{\boldsymbol{p}\sigma}\,, \tag{2}$$

where $c^\dagger_{\mathbf{p}\sigma}$ and $c_{\mathbf{p}\sigma}$ are the creation and annihilation operators of a fermion with momentum $\mathbf{p}$ and spin σ, respectively. In the course of calculating the pair correlation function, expectation values of the form

$$\langle\Phi_0|\,c^\dagger_{\mathbf{p}\sigma}c^\dagger_{\mathbf{q}\sigma'}c_{\mathbf{q}'\sigma'}c_{\mathbf{p}'\sigma}\,|\Phi_0\rangle \tag{3}$$

occur. Compute these expectation values explicitly. What conditions on p, p', q, q' and σ, σ' have to be satisfied so that the amplitudes are nonzero?

b) Next, consider the case $\sigma \neq \sigma'$ and use the above results to calculate explicitly the pair correlation function. **1**pt(s)

c) Finally, consider the interesting case of $\sigma = \sigma'$ and determine the pair-correlation function. Sketch the result. **1**pt(s)

Problem 9.2: Stark Effect for a Harmonic Oscillator [**Oral** | 2 pt(s)]

ID: ex_stark_effect_harmonic_oscillator:aqt2425

Learning objective

The goal of this problem is to apply a static perturbation (as derived in the lecture) up to second order for a simple setup. This setup has the special property that an exact solution exists and therefore the perturbation theory can be tested by a comparison with the exact solution.

The Hamiltonian of a one-dimensional harmonic oscillator within a homogeneous electric field E is given by

$$H = \frac{1}{2}\left(P^2 + Q^2\right) + eEQ\,, \tag{4}$$

where the Hamiltonian is dimensionless, i.e., $[Q, P] = i$. Consider the second term of the Hamiltonian as a perturbation of the free oscillator, that is, $H_1 = Q$ and $\lambda = eE$.

a) Calculate the perturbed eigenfunctions and energy eigenvalues up to second order in λ. **1**pt(s)

b) Compare the result of perturbation theory with the exact solution for the energy of the problem. **1**pt(s)

Problem 9.3: Time-Dependent Perturbation Theory [**Oral** | 2 (+1 bonus) pt(s)]

ID: ex_time_dependent_perturbation_theory:aqt2425

Learning objective

The first subtask is an application of time-dependent perturbation theory in first order. The goal is to identify the correct transition amplitude and evaluate the integrations analytically. The second subtask is a challenging calculation to derive the *exact* solution of the problem—a useful repetition/application of important methods in quantum mechanics. This problem is one of the few examples where the time evolution can be solved analytically (which allows for a verification of perturbation theory).

We investigate a one-dimensional harmonic oscillator with mass m, charge e, and frequency ω in a time-dependent electric field $E(t)$. The Hamiltonian is of the form

$$\begin{aligned} & H = H_0 + H'(t)\,, \\ \text{where} \quad & H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \quad \text{(harmonic oscillator)} \\ \text{and} \quad & H'(t) = ex\,E(t) \quad \text{(perturbation)}\,. \end{aligned} \tag{5}$$

The time-dependency of the external electric field is given by

$$E(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2}\, \cos(\Omega t)\,, \tag{6}$$

where $A \in \mathbb{R}$ is a constant, $\tau > 0$ is a decay rate and $\Omega > 0$ is a frequency.

a) Calculate the transition probability $P_{0\to n}(t, t_0)$ from the ground state $|0\rangle$ at $t_0 \to -\infty$ to an excited state $|n\rangle$ at $t \to +\infty$ in first order perturbation theory. What happens for $\tau \to 0$? **1**pt(s)

Hint: Use $x = \sqrt{\frac{\hbar}{2m\omega}}\,\left(a^\dagger + a\right)$ to evaluate the matrix element.

b) The transition probability can also be calculated *exactly* using the following identity **1**pt(s)

$$\hat{T}\, e^{-i\int_{t_0}^{t} \mathrm{d}t' \left(f(t')a + f^*(t')a^\dagger\right)} = e^{-i\int_{t_0}^{t} \mathrm{d}t' f(t')a}\, e^{-i\int_{t_0}^{t} \mathrm{d}t' f^*(t')a^\dagger}\, e^{\int_{t_0}^{t} \mathrm{d}t' f^*(t') \int_{t_0}^{t'} \mathrm{d}t'' f(t'')} \tag{7}$$

which is a generalization of the well-known relation for the displacement operator. Determine the time evolution for the initial state $|0\rangle$, and show that the transition probabilities $P_{0\to n}(t, t_0)$ for $t_0 \to -\infty$ and $t \to +\infty$ take the form

$$P_{0\to n} = \frac{K^{2n}}{n!} e^{-K^2} \quad \text{with} \quad K = \frac{eA}{2\sqrt{2m\omega\hbar}} e^{-\frac{\tau^2(\omega+\Omega)^2}{4}} \left(1 + e^{\tau^2\omega\Omega}\right) \tag{8}$$

and compare the result with a).

*c) Prove Eq. (7). **+1**pt(s)

Hint: Apply the same method as the proof of the relation $e^{A+B} = e^A\, e^B\, e^{-[A,B]/2}$ requires.