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Problem 7.1: The BCS ground state

[**Written** | 4 pt(s)]

ID: ex_bcs_ground_state:aqt2425

> **Learning objective**
>
> The BCS theory describes conventional superconductors. Here you study the BCS ground state wavefunction as an example for a *quasiparticle vacuum*, i.e., the ground state of a non-interacting fermionic theory. This problem also serves as an exercise for calculations in the formalism of second quantization.

We consider the famous BCS state (named after J. Bardeen, L. N. Cooper, J. R. Schrieffer)

$$|\Omega\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\, c^\dagger_{\boldsymbol{k},\uparrow} c^\dagger_{-\boldsymbol{k},\downarrow} \right) |0\rangle \ , \tag{1}$$

where $u_{\boldsymbol{k}}, v_{\boldsymbol{k}} \in \mathbb{C}$ with $|u_{\boldsymbol{k}}|^2 + |v_{\boldsymbol{k}}|^2 = 1$ and the fermionic operator $c^\dagger_{\boldsymbol{k},\sigma}$ creates a fermion with momentum $\boldsymbol{k}$ and spin $\sigma \in \{\uparrow, \downarrow\}$. The product runs over the Brillouin zone of the lattice (which we do not specify here).

a) Show that the BCS state $|\Omega\rangle$ is normalized. **1**pt(s)

b) Calculate $\langle\Omega|\, c^\dagger_{\mathbf{q},\uparrow} c^\dagger_{-\mathbf{q},\downarrow} \,|\Omega\rangle$ and $\langle\Omega|\, c^\dagger_{\mathbf{q},\sigma} c_{\mathbf{q},\sigma} \,|\Omega\rangle$ for a given wave vector $\boldsymbol{q}$. **1**pt(s)

c) Introduce the new *quasiparticle* operators $\alpha_{\mathbf{k},\sigma}$ via **1**pt(s)

$$\alpha_{\mathbf{k},\uparrow} = u_{\mathbf{k}} c_{\mathbf{k},\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k},\downarrow} \, , \quad \alpha_{-\mathbf{k},\downarrow} = v_{\mathbf{k}} c^\dagger_{\mathbf{k},\uparrow} + u_{\mathbf{k}} c_{-\mathbf{k},\downarrow} \, , \tag{2}$$

Prove that the new operators obey the fermionic anticommutation relations. Show that $\alpha_{\mathbf{k},\sigma} |\Omega\rangle = 0$ for all $\mathbf{k}$ and σ and write down a Hamiltonian for which $|\Omega\rangle$ is the ground state (the *quasiparticle vacuum*).

d) What choice of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ makes $|\Omega\rangle$ the ground state of free fermions (with eigenmodes $c_{\mathbf{k},\sigma}$) ? In this case, what does $\alpha^\dagger_{\mathbf{k},\sigma}$ describe for $|\boldsymbol{k}| \lesseqgtr k_F$ where k_F denotes the Fermi wave vector? **1**pt(s)

Problem 7.2: Coherent states

[**Oral** | 5 pt(s)]

ID: ex_coherent_states_2:aqt2425

> **Learning objective**
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> In this problem you calculate some important properties of *coherent states*, which are quantum states that most closely resemble classical light. Coherent states are ubiquitous in quantum physics and are, for example, an important concept in quantum optics.

A coherent state $|\alpha\rangle$ (sometimes called *Glauber state*) is defined as the right-eigenstate of the (bosonic) annihilation operator a,

$$a\,|\alpha\rangle = \alpha\;|\alpha\rangle\;, \tag{3}$$

with eigenvalue $\alpha \in \mathbb{C}$.

a) Determine the coefficients $c_n(\alpha)$ of the expansion $|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,(\alpha)\;|n\rangle$ for the normalized coherent state, $\langle\alpha|\alpha\rangle = 1$, where $|n\rangle$ is the eigenstate of the occupation operator $n = a^\dagger a$. **1**pt(s)

b) Introduce the displacement operator $D(\alpha) = \exp(\alpha\, a^\dagger - \alpha^*\, a)$ and show that it creates a coherent state $|\alpha\rangle$ when applied to the vacuum $|0\rangle$. **1**pt(s)

Hints: Use the Baker-Campbell-Hausdorff (BCH) formula,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}\,, \tag{4}$$

if the commutator $[A,B]$ commutes with A and B.

c) Calculate the mean particle number $\langle n\rangle = \langle\alpha|\, n\,|\alpha\rangle$ and the variance $(\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2$. To which probability distribution does this correspond? **1**pt(s)

d) Consider a one-dimensional harmonic oscillator with the Hamiltonian $H = \hbar\omega\left(n + \frac{1}{2}\right)$ in the initial state $|\psi(t=0)\rangle = |\alpha_0\rangle$. Show that the time evolution can be written as $|\psi(t)\rangle = e^{i\phi(t)}\,|\alpha(t)\rangle$ with some time-dependent phase $\phi(t)$ and a time-dependent parameter $\alpha(t)$. **1**pt(s)

e) Compute the overlap $\langle\alpha|\alpha'\rangle$ and the operator $\int \mathrm{d}^2\alpha\;|\alpha\rangle\,\langle\alpha|$. Interpret your results. **1**pt(s)

Hints: The integral $\int \mathrm{d}^2\alpha = \int \mathrm{d}(\mathrm{Re}\,\alpha) \int \mathrm{d}(\mathrm{Im}\,\alpha)$ sums over all *complex* values of α. Evaluate this integral in polar coordinates and use the Gamma function $\Gamma(x) = \int_0^\infty \mathrm{d}z\, z^{x-1} e^{-z}$ with $\Gamma(n) = (n-1)!$.

Problem 7.3: Coherent light pulses

[**Oral** | 3 (+1 bonus) pt(s)]

ID: ex_coherent_light_pulses:aqt2425

Learning objective

In Problem 7.2 we introduced coherent states. Here, we use coherent states to describe classical light.

We define the operator

$$a^\dagger = \sum_{\boldsymbol{k},\boldsymbol{\epsilon}} f_{\boldsymbol{k},\boldsymbol{\epsilon}} a^\dagger_{\boldsymbol{k},\boldsymbol{\epsilon}}, \tag{5}$$

where $a^\dagger_{\boldsymbol{k},\epsilon}$ creates a photon with momentum $\boldsymbol{k}$ and polarization $\boldsymbol{\epsilon}$.

a) Calculate the commutator $[a, a^\dagger]$. What is the condition on $f_{\boldsymbol{k},\boldsymbol{\epsilon}}$ such that a and $a^\dagger$ are proper bosonic operators? **1**pt(s)

b) Consider the state $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle$ Calculate the mean values of the electric field $\langle \boldsymbol{E}\rangle$. Does the result correspond to what one expects from a classical system? **1**pt(s)

c) Next, consider a coherent state as defined in Problem 7.2 with $\alpha = 1$. Calculate the mean values of the electric field $\langle \boldsymbol{E}\rangle$. Compare your result with a classical system. **1**pt(s) *d) Let **+1**pt(s)

$$f_{\boldsymbol{k},\epsilon} = \delta_{\boldsymbol{\epsilon},\boldsymbol{e}_x}\delta_{k_x,0}\delta_{k_y,0}\sqrt{\frac{2\pi}{L_z}}\left(\frac{2\sigma}{\pi}\right)^{\frac{1}{4}}\exp(-\sigma(k_z-k_0)^2). \tag{6}$$

Calculate $\langle \boldsymbol{E}(\boldsymbol{r},t)\rangle$ approximately for $k_0\sigma \gg 1$ and exactly using numeric integration for $k_0 = 10$ and $\sigma = 0.1$. Compare your results.

Hints: You may end up with the integrals

$$I_1 = \int_0^\infty dk\sqrt{k}e^{-\sigma(k-k_0)^2}e^{ik(z-t)}, \tag{7}$$

$$I_2 = \int_{-\infty}^0 dk\sqrt{|k|}e^{-\sigma(k-k_0)^2}e^{ik(z+t)}. \tag{8}$$

For $k_0\sigma \gg 1$ the integrands are peaked around $k = k_0$. Thus, the second integral is approximately 0 and in the first integral we can replace $\sqrt{k}$ by $\sqrt{k_0}$ and extend the integration to negative k.