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## Problem 6.1: Properties of bosonic operators

[**Written** | 10 pt(s)]

ID: ex_properties_of_bosonic_operators:aqt2425

**Learning objective**

The first part of this problem reviews properties of bosonic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

In order to solve the harmonic oscillator, one can introduce bosonic creation and annihilation operators satisfying the commutation relation

$$[b, b^\dagger] = 1\,. \tag{1}$$

The occupation number operator is given by $\hat{n} = b^\dagger b$ with eigenstates $|n\rangle$ and eigenvalues $n$.

a) Using (1), show that $b\,|n\rangle$ and $b^\dagger\,|n\rangle$ are eigenstates of $\hat{n}$. **2**pt(s)

b) Prove the following relations: **2**pt(s)

$$b\,|n\rangle = \sqrt{n}\,|n-1\rangle\ , \tag{2}$$

$$b^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\ . \tag{3}$$

c) Show that there has to be a state $|G\rangle$ with $b\,|G\rangle = 0$ and prove that $n$ is an integer. **2**pt(s)
**Hint:** Use the fact that there are no states with negative norm.

d) Using the commutation relations of operators in bosonic Fock space, **2**pt(s)

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0 \tag{4}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} \ldots \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}}\,|0\rangle\,, \tag{5}$$

prove the following relations:

$$b_i\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle\ , \tag{6}$$

$$b_i^\dagger\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}\,|n_1, \ldots, n_i + 1, \ldots\rangle\ . \tag{7}$$

e) Show that the states (5) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum b_i^\dagger b_i$. Now take a Hilbert-space of $m$-modes, i.e., the basis states of the Fock space are **2**pt(s)

$$|n_1, \ldots, n_m\rangle\,.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number $N\,|\Psi\rangle = \hat{n}\,|\Psi\rangle$?

## Problem 6.2: Properties of fermionic operators

[**Written** | 10 pt(s)]

ID: ex_properties_of_fermionic_operators:aqt2425

**Learning objective**

This problem reviews properties of fermionic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

Now we introduce fermionic operators, which fulfill the anti-commutation relation

$$\{a, a^\dagger\} = 1 \,. \tag{8}$$

The occupation number operator is given by $\hat{n} = a^\dagger a$ with eigenstates $|n\rangle$ and eigenvalues $n$.

a) Using (8), show that $a|n\rangle$ and $a^\dagger|n\rangle$ are eigenstates of $\hat{n}$. **2**pt(s)

b) Prove the following relations: **2**pt(s)

$$a|n\rangle = \sqrt{n}\,|1-n\rangle \qquad\qquad a^\dagger|n\rangle = \sqrt{1-n}\,|1-n\rangle \,. \tag{9}$$

c) Show that there has to be a state $|G\rangle$ with $a|G\rangle = 0$ and a state $|H\rangle$ with $a^\dagger|H\rangle = 0$. Further show that these are the only states in the Hilbert space. Assume that $n$ is an integer. **2**pt(s)
**Hint:** Use the fact that there are no states with negative norm.

d) Using the anti-commutation relations of operators in fermionic Fock space, **2**pt(s)

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0 \tag{10}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots (a_i^\dagger)^{n_i} \ldots (a_1^\dagger)^{n_1} |0\rangle \,, \tag{11}$$

prove the following relations:

$$a_i |n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,1}(-1)^{S_i} |n_1, \ldots, n_i - 1, \ldots\rangle \,, \tag{12}$$
$$a_i^\dagger |n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,0}(-1)^{S_i} |n_1, \ldots, n_i + 1, \ldots\rangle \,, \tag{13}$$

where $S_i = n_\infty + \cdots + n_{i+1}$

e) Show that the states (11) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum a_i^\dagger a_i$. Now take a Hilbert-space of $m$-modes, i.e., the basis states of the Fock space are **2**pt(s)

$$|n_1, \ldots, n_m\rangle \,.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number $N|\Psi\rangle = \hat{n}|\Psi\rangle$?

## Problem 6.3: Expectation values of bosonic and fermionic operators [**Oral** | 6 pt(s)]

ID: ex_expectation_values_fock_space:aqt2425

**Learning objective**

Here you will take the basis states of the bosonic and fermionic Fock space and practice the calculation of expectation values of many-particle systems.

In the following, we will work with the operators

$$A = c_i^\dagger c_i \qquad B = c_i^\dagger c_i c_j^\dagger c_j$$
$$C = c_i^\dagger c_j^\dagger c_j c_i \qquad D = c_i^\dagger c_j^\dagger c_i c_j .$$

a) Show that these operators are self-adjoint. Consider the cases $c_i = b_i$ (bosons) and $c_i = a_i$ (fermions). **2**pt(s)

b) Calculate the expectation value of the operators $A$, $B$, $C$ and $D$, taking the states (5), with $c_i = b_i$, and (11), with $c_i = a_i$. **2**pt(s)

c) Finally, determine the matrix element **2**pt(s)

$$\langle m_1, \ldots, m_i, \ldots | \, c_i^\dagger c_j + c_j^\dagger c_i \, | n_1, \ldots, n_i, \ldots \rangle \,,$$

again both for the bosonic and fermionic Fock space and operator algebra.

## Problem 6.4: Time-reversal symmetry

[**Oral** | 10 pt(s)]

ID: ex_time-reversal_symmetry:aqt2425

**Learning objective**

According to Wigner's theorem, symmetry operators fall into two categories: unitary and anti-unitary operators. While symmetries are mostly described by unitary operators (e.g. $U(1)$ symmetry, rotational symmetry, translation symmetry), time-reversal symmetry is a fundamental (discrete) symmetry that is represented by an anti-unitary operator. In this problem, you will study the physics of time reversal for simple examples.

In the following, we consider a system with a time-independent Hamiltonian $H$ that is invariant under time reversal given by the operator $T$. Since $T$ is connected to a symmetry of the system, it commutes with the Hamiltonian, $[H, T] = 0$. The transformation of the time evolution operator $U(t)$ under time reversal is given by

$$T^{-1} U(t) T = U(-t) \,. \tag{14}$$

a) Show by using $U(t) = \exp(-\frac{i}{\hbar} H t)$ that $T$ is an anti-linear operator. Since $T$ is anti-linear, Wigner's theorem implies that $T$ is an anti-unitary operator. **2**pt(s)

Show further that if $|\psi\rangle$ is a solution of the Schrödinger equation, $T|\psi\rangle$ is a solution of the Schrödinger equation with $t \to -t$. Thus, $T|\psi\rangle$ satisfies the equation $-i\hbar\partial_t T|\psi\rangle = HT|\psi\rangle$.

**Hint:** An anti-linear operator has the property that $T(c|v\rangle) = c^* T|v\rangle$ for $c \in \mathbb{C}$ and $|v\rangle \in \mathcal{H}$ with some Hilbert space $\mathcal{H}$.

b) For spinless particles, the time-reversal operator $T$ in the position basis satisfies **2**pt(s)

$$T|x\rangle = |x\rangle \,. \tag{15}$$

Show that $T\psi(x) = \psi^*(x)$. In order to do so, consider the action of $T$ on some arbitrary state $|\psi\rangle$ and use $\psi(x) = \langle x|\psi\rangle$. It thus follows that in the position representation for spinless particles, $T = K$, where $K$ denotes the complex conjugation with $Kc = c^* K$ for $c \in \mathbb{C}$. Show that consequently for spinless particles $T^2 = 1$.

c) Derive the transformation laws for the position, momentum and angular momentum operator in the position representation. How does this connect to classical physics? Show that the system is time-reversal invariant if the Hamiltonian is real, i.e. $H^* = H$. **2**pt(s)

d) Consider now a spin-1/2 particle. In this case, the time-reversal operator can be written as **2**pt(s)

$$T = \exp\left(-i\frac{\pi}{2}\sigma_y\right) K = -i\sigma_y K\,, \tag{16}$$

where $\sigma_y$ is a Pauli matrix. Derive the transformation of the spin $\mathbf{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)^T$ under the transformation (16) and show that $T^2 = -I$, where $I$ is the identity operator.

e) Show that in a system that is time-reversal invariant and $T^2 = -I$ (as for example for a spin-1/2 particle), all energy levels are (at least) doubly degenerate. This is known as *Kramers' theorem*. **2**pt(s)