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Problem 11.1: Time-Dependent Perturbation Theory

[**Oral** | 2 (+1 bonus) pt(s)]

ID: ex_time_dependent_perturbation_theory:aqt2425

Learning objective

The first subtask is an application of time-dependent perturbation theory in first order. The goal is to identify the correct transition amplitude and evaluate the integrations analytically. The second subtask is a challenging calculation to derive the *exact* solution of the problem—a useful repetition/application of important methods in quantum mechanics. This problem is one of the few examples where the time evolution can be solved analytically (which allows for a verification of perturbation theory).

We investigate a one-dimensional harmonic oscillator with mass m, charge e, and frequency ω in a time-dependent electric field $E(t)$. The Hamiltonian is of the form

$$\begin{aligned} & H = H_0 + H'(t)\,, \\ \text{where} \quad & H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \quad \text{(harmonic oscillator)} \\ \text{and} \quad & H'(t) = ex\,E(t) \quad \text{(perturbation)}\,. \end{aligned} \tag{1}$$

The time-dependency of the external electric field is given by

$$E(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2}\, \cos(\Omega t)\,, \tag{2}$$

where $A \in \mathbb{R}$ is a constant, $\tau > 0$ is a decay rate and $\Omega > 0$ is a frequency.

a) Calculate the transition probability $P_{0\to n}(t, t_0)$ from the ground state $|0\rangle$ at $t_0 \to -\infty$ to an excited state $|n\rangle$ at $t \to +\infty$ in first order perturbation theory. What happens for $\tau \to 0$? **1pt(s)**

Hint: Use $x = \sqrt{\frac{\hbar}{2m\omega}}\,\left(a^\dagger + a\right)$ to evaluate the matrix element.

b) The transition probability can also be calculated *exactly* using the following identity **1pt(s)**

$$\hat{T}\, e^{-i\int_{t_0}^{t} \mathrm{d}t'\, \left(f(t')a + f^*(t')a^\dagger\right)} = e^{-i\int_{t_0}^{t} \mathrm{d}t' f(t')a}\, e^{-i\int_{t_0}^{t} \mathrm{d}t' f^*(t')a^\dagger}\, e^{\int_{t_0}^{t} \mathrm{d}t' f^*(t')\int_{t_0}^{t'} \mathrm{d}t'' f(t'')} \tag{3}$$

which is a generalization of the well-known relation for the displacement operator. Determine the time evolution for the initial state $|0\rangle$, and show that the transition probabilities $P_{0\to n}(t, t_0)$ for $t_0 \to -\infty$ and $t \to +\infty$ take the form

$$P_{0\to n} = \frac{K^{2n}}{n!} e^{-K^2} \quad \text{with} \quad K = \frac{eA}{2\sqrt{2m\omega\hbar}} e^{-\frac{\tau^2(\omega+\Omega)^2}{4}} \left(1 + e^{\tau^2\omega\Omega}\right) \tag{4}$$

and compare the result with a).

*c) Prove Eq. (3). **+1**pt(s)

Hint: Apply the same method as the proof of the relation $e^{A+B} = e^A \, e^B \, e^{-[A,B]/2}$ requires.

Problem 11.2: Spontaneous decay of the Hydrogen Atom [**Written** | 3 pt(s)]

ID: ex_spontaneous_decay_hydrogen_atom:aqt2425

> **Learning objective**
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> In this exercise you will calculate which transitions in the Hydrogen atom can occur spontaneously. Based on this you then will calculate the transition rates for electrons from the $n = 2$ manifold back to the ground state.

Consider an excited Hydrogen atom in the state $|n, l, m\rangle$ inside the electromagnetic vacuum. We try to find the possible transitions into a state $|n', l', m'\rangle$ by emitting a photon in the mode $|n_{k,\lambda} = 1\rangle$. Assuming the light-matter interaction is adiabatically turned on and off we can calculate the transition in first order perturbation theory as

$$\langle n', l', m'; n_{k,\lambda} = 1|\Psi(t)\rangle = \frac{1}{i\hbar} e^{-i\mathcal{E}_f(t-t_0)} \int_{t_0}^{t} dt_1 \, \langle n', l', m'; n_{k,\lambda} = 1| \, H_{\text{int}}(t_1) \, |n, l, m; n_{k,\lambda} = 0\rangle \, . \quad (5)$$

In the lecture It was shown that this results in the transition rate

$$\Gamma = \frac{d}{dt} | \, \langle n', l', m'; n_{k,\lambda} = 1|\Psi(t)\rangle \, |^2 = \frac{2\alpha\omega}{3} \frac{(\hbar\omega)^2}{mc^2 E_{\text{R}}} |r_{ab}/a_{\text{B}}|^2,$$

where E_{R} is the Rydberg energy, a_{B} the Bohr radius and ω is the frequency resonant to the transition $E_n \to E_{n'}$. Further, the dipole matrix element $\boldsymbol{r}_{ab}$ is given by

$$\boldsymbol{r}_{ab} = \langle n', l', m'| \, \boldsymbol{r} \, |n, l, m\rangle \, . \quad (6)$$

a) First rewrite $\boldsymbol{r} = (x, y, z)^T = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$. Then express this vector in terms of the spherical harmonics $Y_{l,m}(\theta, \phi)$. Use this to find the transitions which are allowed in first order. **1**pt(s)

b) Explicitly calculate the dipole matrix elements for the $n = 2 \to n = 1$ transition. **1**pt(s)

c) For an Hydrogen atom prepared in any of the $n = 2$ states, what are the average life times? **1**pt(s)

Problem 11.3: Squeezed States [**Oral** | 3 pt(s)]

ID: ex_squeezed_states:aqt2425

> **Learning objective**
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> In Problem Set 7, we have studied coherent states. The quantum state of the harmonic oscillator, that minimizes the uncertainty relation with the uncertainty equally distributed between the non-commuting observables Q and P, is such a coherent state. In this exercise, we study *squeezed states*, for which the uncertainty of one of the observables is smaller than for the coherent state. To respect the uncertainty principle, the uncertainty of the other observable must be larger than for the coherent state. Squeezed

states of light are an example of non-classical light.

For the harmonic oscillator with frequency ω, coherent states obey

$$\langle \Delta Q^2 \rangle = \frac{\hbar}{2m\omega} \quad \text{and} \quad \langle \Delta P^2 \rangle = \frac{\hbar m \omega}{2} \tag{7}$$

and thus squeezed states fulfill

$$\langle \Delta Q^2 \rangle \leq \frac{\hbar}{2m\omega} \quad \text{or} \quad \langle \Delta P^2 \rangle \leq \frac{\hbar m \omega}{2} \,. \tag{8}$$

We define the squeeze operator $S(\epsilon)$ as

$$S(\epsilon) = \exp\left(\frac{\epsilon^*}{2}\, a^2 - \frac{\epsilon}{2}\left(a^\dagger\right)^2\right) \,. \tag{9}$$

a) Write the operators Q and P in terms of the creation and annihilation operators of the harmonic oscillator and prove that coherent states satisfy equation (7). **1**pt(s)

b) Derive the following relations for $S(\epsilon)$ with $\epsilon = r\ \exp(i\phi)$, where r is an arbitrary radius and ϕ an arbitrary phase: **1**pt(s)

$$S^\dagger(\epsilon) = S^{-1}(\epsilon) = S(-\epsilon) \,, \tag{10}$$
$$S^\dagger(\epsilon)\ a\, S(\epsilon) = a\ \cosh(r) - a^\dagger\, e^{i\phi}\ \sinh(r) \,, \tag{11}$$
$$S^\dagger(\epsilon)\ a^\dagger\, S(\epsilon) = a^\dagger\ \cosh(r) - a\, e^{-i\phi}\ \sinh(r) \,. \tag{12}$$

c) Verify that the states $|\alpha\,,\,\epsilon\rangle = D(\alpha)\ S(\epsilon)\ |0\rangle$ are squeezed states. Here, $D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right)$ is the displacement operator. **1**pt(s)