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Problem 6.1: System of three interacting spin-1/2 particles [**Oral** | 4 pt(s)]

ID: ex_system_three_interacting_spin_one_half_particles:aqt2324

Learning objective

In this exercise you will be going through an instructive example of a few particle spin system. The task is to show that using the theorem of addition of angular momentum, it is possible to analytically determine the ground state of three spins coupled antiferromagnetically.

Let us consider a system composed of three spin-1/2 particles.

a) What is the dimension of the Hilbert space? **2**pt(s)

The total spin operator can be defined as $\mathbf{S} = \sum_{i=1}^{3} \mathbf{S}^{(i)}$ and its z projection as $S_z = \sum_{i=1}^{3} S_z^{(i)}$. What are the eigenvalues and eigenstates of $\mathbf{S}^2$ and S_z? Express the states in the spin-1/2 basis $\left|S_z^{(1)}, S_z^{(2)}, S_z^{(3)}\right\rangle$.

b) The Hamiltonian of the system is **2**pt(s)

$$H = J \sum_{i=1}^{3} \mathbf{S}^{(i)} \cdot \mathbf{S}^{(i+1)}, \qquad J > 0.$$

Here we assume a periodic system (for $i = 3$ take $i + 1 = 1$). Calculate the eigenstates and eigenenergies of this Hamiltonian.

Hint: Rewrite H as a function of S^2 and $S^{(i)2}$.

Problem 6.2: Perturbing the harmonic oscillator with a weak electric field [**Oral** | 6 pt(s)]

ID: ex_charged_oscillator_in_electric_field_stationary_perturbation:aqt2324

Learning objective

We revisit the problem of a charged oscillating particle under the influence of an electric field. Although this problem was solved exactly previously, it is instructive to see that stationary perturbation theory produces consistent results.

A particle of charge q and mass m, which is moving in a one-dimensional harmonic potential of frequency ω, is subject to a *weak* electric field $\mathcal{E}$ in the x-direction. As we have seen, the Hamiltonian governing the interaction between the oscillating charge and the external electric field can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_p = -\frac{\hbar}{2m}\frac{d^2}{dX^2} + \frac{1}{2}m\omega^2 X^2 + q\mathcal{E}X. \tag{1}$$

a) First, consider the general theory of nondegenerate stationary perturbation theory. **2**pt(s)

(i) Explain and write down the equations on how you would calculate corrections for the eigenvalues and eigenfunctions up to second and first order, respectively.

(ii) State briefly the necessary condition over the parameter λ for the convergence of the expansion of the perturbed eigenvalues E_n^i and eigenfunctions $\left|\psi_n^{(1)}\right\rangle$ in

$$\begin{aligned} E_n &= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots\,, \\ |\psi_n\rangle &= |\phi_n\rangle + \lambda\left|\psi_n^{(1)}\right\rangle + \lambda^2\left|\psi_n^{(2)}\right\rangle + \cdots\,, \end{aligned} \tag{2}$$

where $\hat{H}_0\,|\phi_n\rangle = E_n^{(0)}\,|\phi_n\rangle$ and $\hat{H}_p = \lambda\hat{W}$.

As a consistency check, what happens to this convergence condition if there are unperturbed energy levels E_n^0 which are degenerate?

Hint: Identify the expression for λ by comparing the equations you obtain in (i) with (2).

b) Use the previously explained formalism to calculate the first nonzero energy correction to (1), and compare it with the exact result obtained in exercise 2.2 in problem sheet 2. **2**pt(s)

c) Calculate the correspondent corrections to the eigenfunctions up to first order in perturbation theory. **2**pt(s)

Problem 6.3: Time-reversal symmetry

[**Written** | 10 pt(s)]

ID: ex_time-reversal_symmetry:aqt2324

Learning objective

According to Wigner's theorem, symmetry operators fall into two categories: unitary and anti-unitary operators. While symmetries are mostly described by unitary operators (e.g. $U(1)$ symmetry, rotational symmetry, translation symmetry), time-reversal symmetry is a fundamental (discrete) symmetry that is represented by an anti-unitary operator. In this problem, you will study the physics of time reversal for simple examples.

In the following, we consider a system with a time-independent Hamiltonian H that is invariant under time reversal given by the operator T. Since T is connected to a symmetry of the system, it commutes with the Hamiltonian, $[H, T] = 0$. The transformation of the time evolution operator $U(t)$ under time reversal is given by

$$T^{-1}U(t)T = U(-t)\,. \tag{3}$$

a) Show by using $U(t) = \exp(-\frac{i}{\hbar}Ht)$ that T is an anti-linear operator. Since T is anti-linear, Wigner's theorem implies that T is an anti-unitary operator. **2**pt(s)

Show further that if $|\psi\rangle$ is a solution of the Schrödinger equation, $T\,|\psi\rangle$ is a solution of the Schrödinger equation with $t \to -t$. Thus, $T\,|\psi\rangle$ satisfies the equation $-i\hbar\partial_t T\,|\psi\rangle = HT\,|\psi\rangle$.

Hint: An anti-linear operator has the property that $T(c\,|v\rangle) = c^*T\,|v\rangle$ for $c \in \mathbb{C}$ and $|v\rangle \in \mathcal{H}$ with some Hilbert space $\mathcal{H}$.

b) For spinless particles, the time-reversal operator T in the position basis satisfies **2**pt(s)

$$T|x\rangle = |x\rangle \ . \tag{4}$$

Show that $T\psi(x) = \psi^*(x)$. In order to do so, consider the action of T on some arbitrary state $|\psi\rangle$ and use $\psi(x) = \langle x|\psi\rangle$. It thus follows that in the position representation for spinless particles, $T = K$, where K denotes the complex conjugation with $Kc = c^*K$ for $c \in \mathbb{C}$. Show that consequently for spinless particles $T^2 = 1$.

c) Derive the transformation laws for the position, momentum and angular momentum operator in the position representation. How does this connect to classical physics? Show that the system is time-reversal invariant if the Hamiltonian is real, i.e. $H^* = H$. **2**pt(s)

d) Consider now a spin-1/2 particle. In this case, the time-reversal operator can be written as **2**pt(s)

$$T = \exp\left(-i\frac{\pi}{2}\sigma_y\right) K = -i\sigma_y \, K \,, \tag{5}$$

where σ_y is a Pauli matrix. Derive the transformation of the spin $\mathbf{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)^T$ under the transformation (5) and show that $T^2 = -I$, where I is the identity operator.

e) Show that in a system that is time-reversal invariant and $T^2 = -I$ (as for example for a spin-1/2 particle), all energy levels are (at least) doubly degenerate. This is known as *Kramers' theorem.* **2**pt(s)