Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

November 15th, 2023
WS 2023/24

Problem 5.1: Angular momentum commutation relations [**Oral** | 7 pt(s)]

ID: ex_j_angularmo_relations:aqt2324

> **Learning objective**
>
> In this exercise, you will prove the commutation relations that were stated in the lecture.

A generalized angular momentum $\boldsymbol{J} = (J_x, J_y, J_z)$ operator has the following property

$$[J_k, J_l] = i\hbar\epsilon_{klm} J_m \quad k, l, m = x, y, z. \tag{1}$$

If $\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x + iJ_y = J_\mp^\dagger$, show the following:

a) $[J_z, J_\pm] = \pm\hbar J_\pm$ **2** pt(s)

b) $[J_+, J_-] = 2\hbar J_z$ **1** pt(s)

c) $J_\pm J_\mp = \boldsymbol{J}^2 - J_z(J_z \mp \hbar)$ **2** pt(s)

d) $[\boldsymbol{J}^2, J_\pm] = 0$ **2** pt(s)

Problem 5.2: Clebsch-Gordan coefficients and spin-orbit coupling [**Oral** | 4 (+2 bonus) pt(s)]

ID: ex_clebsch_gordan_coefficients_spin_orbit_coupling:aqt2324

> **Learning objective**
>
> In this problem you will apply the angular momentum addition theorem. As an important use case, we consider the spin-orbit coupling in the hydrogen atom, which is the leading relativistic correction.

The spin-orbit coupling between the electron's spin $\boldsymbol{S}$ and the orbital angular momentum $\mathbf{L}$ for a hydrogen atom is given by the Hamiltonian

$$H_{\text{LS}} = f(r)\, \boldsymbol{L} \cdot \boldsymbol{S} = f(r) \sum_{\alpha=x,y,z} L_\alpha \otimes S_\alpha\,, \tag{2}$$

where $f(r) = e^2/2m_e^2c^2r^3$. The spin-orbit coupling can be seen as a perturbation to the non-relativistic Hamiltonian $H_0 = \boldsymbol{P}^2/2m - e^2/r$ of the hydrogen atom.

a) Define the total angular momentum operator as **2** pt(s)

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{L} \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{S} \tag{3}$$

and show that $\boldsymbol{J}^2$ and J_z commute both with H_0 and H_{LS}.

b) Consider the subspace with orbital angular momentum ℓ and spin s. We can write the eigenstates $|j,m\rangle$ of $\boldsymbol{J}^2$ and J_z as linear combinations of L_z- and S_z-eigenstates $|m_\ell, m_s\rangle = |\ell, m_\ell\rangle \otimes |s, m_s\rangle$, **2pt(s)**

$$|j,m\rangle = \sum_{m_\ell, m_s} c(m_\ell, m_s; j, m)\, |m_\ell, m_s\rangle \,. \tag{4}$$

The coefficients c are called *Clebsch-Gordan coefficients*. Due to their ubiquity in quantum physics there are comprehensive tables available, e.g.,

`http://pdg.lbl.gov/2011/reviews/rpp2011-rev-clebsch-gordan-coefs.pdf`.

Use this table to write down the change of basis (4) in the subspace with $\ell = 1$ and $s = 1/2$ explicitely.

*c) Derive the Clebsch-Gordan coefficients in b) by hand. **+2pt(s)**

Hint: Start with the *stretched state* $|j = 3/2, m_j = 3/2\rangle$ and use the ladder operator $J_- = J_x - iJ_y$ which acts as

$$J_- \,|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j, m-1\rangle \;. \tag{5}$$

Problem 5.3: Bound states of a spherical potential well

[**Written** | 6 pt(s)]

ID: ex_bound_states_spherical_potential_well:aqt2324

Learning objective

Here we will derive the bound states of a spherically symmetric potential well. To do so, we will exploit the rotation symmetry of the problem and show that the radial solutions are given by *spherical Bessel functions* (see lecture). The idea is to explicitly derive the transcendental equation that determines the eigenenergies for bound states with angular momentum $l = 0$.

Consider the Hamiltonian of a particle in three dimensions

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}) \tag{6}$$

with the spherically symmetric and piecewise constant potential

$$V(\boldsymbol{r}) = V(r) = \begin{cases} 0 & r > R \\ V_0 & r \leq R \end{cases} \tag{7}$$

with $r = |\boldsymbol{r}|$, $R > 0$ the radius of the potential well and $V_0 < 0$ the potential depth.

Your goal is to find the bound states and eigenenergies of this system and the conditions that are necessary for their existence.

a) Make the separation ansatz $\Psi(\boldsymbol{r}) = R_l(r) \cdot Y_{lm}(\theta, \varphi)$ with spherical harmonics Y_{lm} and show that the eigenvalue problem reduces to **2pt(s)**

$$\left[\rho^2 \partial_\rho^2 + 2\rho\partial_\rho + \rho^2 - l(l+1)\right] \tilde{R}_l(\rho) = 0 \tag{8}$$

with $\rho \equiv K_r r$ and $\tilde{R}_l(\rho) \equiv R_l(r)$ where $K_r \equiv \sqrt{\frac{2m(E-V(r))}{\hbar^2}}$.

b) Write down the general solution of the radial problem in the two regions $r > R$ and $r \leq R$ for a given angular momentum l and formulate the continuity and boundary conditions that the eigenstates must satisfy. **2**pt(s)

Hint: Use that the solutions of the differential equation

$$\left[x^2 \partial_x^2 + 2x\partial_x + x^2 - l(l+1)\right] y(x) = 0 \tag{9}$$

are given by the *spherical Bessel functions*

$$j_l(x) = (-x)^l \left(\frac{1}{x}\partial_x\right)^l \frac{\sin(x)}{x} \quad \text{and} \quad y_l(x) = -(-x)^l \left(\frac{1}{x}\partial_x\right)^l \frac{\cos(x)}{x} \tag{10}$$

for $l \in \mathbb{N}_0$. (The functions y_l are sometimes denoted n_l and referred to as *spherical Neumann functions*.) Write the eigenstates in terms of these functions.

c) Consider the simplest case for $l = 0$. Find explicit expressions for the bound states and derive a transcendental equation to determine their eigenenergies. At which potential depth V_0 appears the first bound state? **2**pt(s)

Hint: Use your knowledge of the one-dimensional potential well to analyze the transcendental equation.