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Problem 3.1: Translation Operator

[**Oral** | 10 pt(s)]

ID: ex_translation_qm:aqt2324

Learning objective

The main idea behind this exercise is to get comfortable with the translation operator. As we will see in the next problem, this will be quite useful. Here, we take a complementary approach to how these operators are presented in the lectures.

Let $\hat{T}$ be the translation operator. It may be defined by

$$\hat{T}(a)|x\rangle = |x+a\rangle \tag{1}$$

where $|x\rangle$ is an eigenstate of the position operator and $a \in \mathbb{R}$.

a) Show that $\hat{T}(a)$ has the following effect on the wave function: **2**pt(s)

$$\hat{T}(a)\,\psi(x) = \psi(x-a)\,. \tag{2}$$

b) Show that the translation operator commutes with the momentum operator. In order to accomplish this you should introduce the Taylor expansion of $\psi(x-a)$ near x. **2**pt(s)

c) Show that $\hat{T}(a)$ is unitary, i.e., $\hat{T}(a)^\dagger\,\hat{T}(a) = \mathbb{I}$. What are the eigenvalues and eigenvectors of $\hat{T}(a)$? Moreover show that it is sufficient to know the wavefunction on an interval length a. To proof this determine the eigenvalue λ_a of **2**pt(s)

$$\hat{T}(a)\,\Phi(x) = \lambda_a \Phi(x) = \Phi(x-a) \tag{3}$$

where Φ_x is an eigenstate of the translation operator.

Hint: Use the Taylor expansion of $T(a)$ to determine λ_a.

d) Proof the subsequent identity for the position operator $\hat{x}$: **2**pt(s)

$$\hat{T}(a)\,\hat{x}\hat{T}(a)^\dagger = \hat{x} - a\mathbb{I} \tag{4}$$

e) Calculate the commutator $[\hat{x}, \hat{T}(a)]$. **2**pt(s)

Problem 3.2: Application of the translation operator

[**Written** | 6 pt(s)]

ID: ex_ionic_potential_1d_translation:aqt2324

> **Learning objective**
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> In this exercise you will investigate an application of the use of translation operator in the context of solid state physics, where regularly spaced atoms are under the influence of a potential giving rise to band-gaps.

Atoms are arranged in one dimension, with spacing a. The ionic potential is modeled by δ- functions and strength $V_0 > 0$, giving the one-particle Hamiltonian

$$H = T + V(x) = \frac{p^2}{2m} + aV_0 \sum_n \delta(x - na). \tag{5}$$

a) Solve the equation above in the interval $0 < x < a$. Now use the periodicity of the atomic potential, namely the eigenvalue equation of the translation operator $\Psi(x') = \Psi(x + na) = T(-na)\Psi(x) = e^{iqna}\Psi(x)$ to show that candidate wave functions are of the form **2**pt(s)

$$\Psi(x'(x)) = A_0 e^{i(q-k)na} e^{ikx} + B_0 e^{i(q+k)na} e^{-ikx}. \tag{6}$$

What are k and q?

Note: "candidate wave functions" in this context means that we can use this wave function at an arbitrary $x' = x + na$, where $n \in \mathbb{Z}$, i.e. that we have a wave function for the whole system. In this case we have constructed one with the help of the translation operator based on the solutions that we have calculated for the interval $0 < x < a$.

b) So far we have ignored the locations of the δ functions. Now we deal with these missing positions by constraining the ansatz Eq. 6 appropriately: At position $x = na$, formulate (dis-) continuity constraints on the (derivative) wave function. Use these requirements to build a set of equations for A_0 and B_0, and show the solution of these equations can be formulated as: **2**pt(s)

$$\cos qa = \alpha \frac{\sin \beta}{\beta} + \cos \beta, \tag{7}$$

where $\alpha = ma^2V_0/\hbar^2$ and $\beta = \sqrt{2mEa^2/\hbar^2}$.

c) Show that there is a non-trivial solution for certain energy ranges. Plot the lowest few of the resulting bands. **2**pt(s)

Problem 3.3: Coherent states

[**Oral** | 8 (+2 bonus) pt(s)]

ID: ex_coherent_states:aqt2324

> **Learning objective**
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> The eigenstates of the harmonic oscillator Hamiltonian $|n\rangle$, are not eigenstates of the ladder operators. The coherent state which is an eigenstate of the annihilation operator is a useful object for example in quantum optics. In this exercise you investigate some properties of coherent states and realize their

convenience especially when describing the dynamic behavior of a quantum harmonic oscillator.

For every $\alpha \in \mathbb{C}$, we define the coherent state

$$|\psi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n \geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{8}$$

being $|n\rangle$ an eigenstate of the 1-dimensional harmonic oscillator Hamiltonian.

a) Verify that the coherent state $|\psi_\alpha\rangle$ is an eigenstate of the annihilation operator where α is the eigenvalue, then show that the creation operator has no eigenstates. **2pt(s)**

b) Show that **2pt(s)**

$$|\psi_\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle \tag{9}$$

where $a^\dagger$ and a are creation and annihilation operators, respectively.
(Tip: $e^{-\alpha a} |0\rangle = |0\rangle$)

*c) Calculate $\langle x \rangle_\alpha = \langle \psi_\alpha | x | \psi_\alpha \rangle$, $\langle p \rangle_\alpha$, Δx_α, Δp_α and show that, for all $\alpha \in \mathbb{C}$, **+2pt(s)**

$$\Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2} \tag{10}$$

is valid, *i.e.*, coherent states minimize the position and momentum uncertanty relation. (Tip: $a |\psi_\alpha\rangle = \alpha |\psi_\alpha\rangle$)

d) Show that coherent states are not orthogonal and the relation **2pt(s)**

$$\langle \psi_\alpha | \psi_\beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^* \beta)} \tag{11}$$

e) Derive the time evolution of coherent state $|\psi_\alpha(t)\rangle$ and of the expectation values $\langle x \rangle_t$ and $\langle p \rangle_t$ under the Hamiltonian $H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$. **2pt(s)**