Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

October 25th, 2023
WS 2023/24

Problem 2.1: Time evolution of a three-level quantum mechanical system [**Oral** | 5 pt(s)]

ID: ex_3_state_qm_time_dependency:aqt2324

Learning objective

This exercise illustrates how to obtain expected values from typical observables in quantum mechanical systems with their corresponding probabilities. Additionally, we can also see how these values change as a function of time.

We consider a system with a three-dimensional Hilbert space, which is defined by the orthonormal basis $|1\rangle, |2\rangle, |3\rangle$. In this basis the Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \hbar\omega(|1\rangle\langle 1| + 2|2\rangle\langle 2| + 2|3\rangle\langle 3|) = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{1}$$

with a constant frequency ω. Assume that at a time $t = 0$ the system is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle. \tag{2}$$

a) If we measure the energy of the system at $t = 0$, which values can we measure and with which probabilities do they occur? Calculate the expectation value of $\langle \hat{H} \rangle$ and the variance $\Delta \hat{H}$. **1**pt(s)

b) Still at $t = 0$, if we wanted to obtain a measurement of observables $\hat{A}$ and $\hat{B}$ given by **1**pt(s)

$$\hat{A} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{B} = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

which values would we be able to find? What are the associated probabilities?

c) Now we allow the system to evolve in time, i.e., $t > 0$. Calculate the state $|\psi(t)\rangle$. **1**pt(s)

d) How do the expected values of $\hat{A}$ and $\hat{B}$ evolve over time? In other words, calculate $\langle \hat{A} \rangle (t)$ $\langle \hat{B} \rangle (t)$ and make a simple plot (or sketch) of these as a function of time. **1**pt(s)

e) What result do you obtain if you make a measurement of the observables $\hat{A}$ and $\hat{B}$ at a time t? What are the maximum and minimum values of these measurements for them? **1**pt(s)

Problem 2.2: Harmonic oscillator in an electric field [**Oral** | 4 pt(s)]

ID: ex_harmonic_oscillator_linear_term:aqt2324

> **Learning objective**
>
> Here we will solve the Harmonic oscillator in presence of an external perturbation - electric field. Problems as such are often discussed in the context of perturbation theory. However, this problem can be solved exactly, without resorting to perturbative methods.

Consider a harmonic oscillator with mass m, frequency ω and momentum $\hat{P}$ and position operators $\hat{Q}$ in presence of a static electric field E acting on charge q. The Hamiltonian then is given by

$$H = H_0 + Eq\hat{Q}, \quad H_0 = \frac{\hat{P}^2}{2m} + \frac{m\omega^2\hat{Q}^2}{2}. \tag{4}$$

a) As a first step, diagonalize the Hamiltonian H_0 by introducing the creation and annihilation operators (see lecture notes), and obtain the energy eigenvalues and wavefunctions. **1**pt(s)

b) Now in order to solve the full problem, rewrite H to $H = \alpha\hat{P}^2 + \beta(\hat{Q} + \gamma)^2 + \delta$. Find the corresponding expressions for $\alpha, \beta, \gamma, \delta$. **1**pt(s)

c) Finally, write down the energies and the wavefunctions for the Hamiltonian obtained above. **2**pt(s)

Problem 2.3: Harmonic oscillator in the Heisenberg picture [**Written** | 4 (+2 bonus) pt(s)]

ID: ex_harmonic_oscillator_Heisenberg_picture:aqt2324

> **Learning objective**
>
> In this problem you will use the Heisenberg formulation of quantum mechanics to compute the position and momentum expectation values of a one dimensional harmonic oscillator. Using the relation to the familiar operators in the Schrödinger picture (eq. 5), you will derive the new ladder operators ($a_H(t)$, $a_H^\dagger(t)$) and the position and momentum operators ($Q_H(t)$, $P_H(t)$) in the Heisenberg picture which will now depend on time. One goal is to remark the connection between the equations of motion for operators in the Heisenberg picture and Hamilton's equations in classical mechanics.

The operators in the Heisenberg picture and operators in the Schrödinger picture are related through

$$A_H(t) = U^{-1}(t) A_S U(t) \quad \text{with} \quad U(t) = e^{-\frac{i}{\hbar}Ht}. \tag{5}$$

$U(t)$ is the time evolution operator which obeys the Schrödinger equation $i\hbar\partial_t U(t) = HU(t)$. We define the time-independent states in the Heisenberg picture as $|\psi_H\rangle := |\psi_S(t=0)\rangle$. The index H stands for operators and states in the *Heisenberg* picture while the index S refers to the *Schrödinger* picture.

a) Derive the Heisenberg equation of motion for operators, **1**pt(s)

$$\partial_t A_H(t) = \frac{1}{i\hbar}[A_H(t), H], \tag{6}$$

starting from Eq. (5). How does this relation change if A_S also depends on time (see (HBG) formula on lecture notes)?

b) We now consider the Hamiltonian for a harmonic oscillator **1**pt(s)

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2 = \hbar\omega\left(a_S^\dagger a_S + \frac{1}{2}\right), \quad [a_S, a_S^\dagger] = 1. \tag{7}$$

Show that

$$a_H(t) = e^{-i\omega t}a_S, \qquad a_H^\dagger(t) = e^{i\omega t}a_S^\dagger. \tag{8}$$

Hint: Let $a_H(t)$ act onto an eigenstate of the harmonic oscillator or use the Baker-Campbell-Hausdorff formula.

Intermediate descriptions for next block of tasks.

*c) Use the results of the previous task to derive the relations **+2**pt(s)

$$Q_H(t) = Q_H(t; P_S, Q_S), \qquad P_H(t) = P_H(t; P_S, Q_S). \tag{9}$$

d) Show for the harmonic oscillator that the Heisenberg equations of motion (6) lead to Hamilton's classical equations of motion for the operators $Q_H(t)$ and $P_H(t)$. **1**pt(s)

e) We define at time $t = 0$ the state $|\psi_S(t=0)\rangle = |\psi_H\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) =: |\gamma\rangle$ (in the basis of the number operator $N|n\rangle = n|n\rangle$). Compute the expectation values **1**pt(s)

$$\langle Q_H(t)\rangle_\gamma = \langle\gamma|Q_H(t)|\gamma\rangle, \qquad \langle P_H(t)\rangle_\gamma = \langle\gamma|P_H(t)|\gamma\rangle, \tag{10}$$

for arbitrary times t.