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Information

This sheet only contains written exercises and no oral ones. Please **hand in your results in the lecture** on thursday the 22.12.2022.

Problem 9.1: Time-Dependent Perturbation Theory

[**Written** | 2 (+1 bonus) pt(s)]

ID: ex_time_dependent_perturbation_theory:aqt2223

Learning objective

The first subtask is an application of time-dependent perturbation theory in first order. The goal is to identify the correct transition amplitude and evaluate the integrations analytically. The second subtask is a challenging calculation to derive the *exact* solution of the problem—a useful repetition/application of important methods in quantum mechanics. This problem is one of the few examples where the time evolution can be solved analytically (which allows for a verification of perturbation theory).

We investigate a one-dimensional harmonic oscillator with mass m, charge e, and frequency ω in a time-dependent electric field $E(t)$. The Hamiltonian is of the form

$$\begin{aligned} & H = H_0 + H'(t)\,, \\ \text{where} \quad & H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \quad \text{(harmonic oscillator)} \\ \text{and} \quad & H'(t) = ex\,E(t) \quad \text{(perturbation)}\,. \end{aligned} \tag{1}$$

The time-dependency of the external electric field is given by

$$E(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2}\, \cos(\Omega t)\,, \tag{2}$$

where $A \in \mathbb{R}$ is a constant, $\tau > 0$ is a decay rate and $\Omega > 0$ is a frequency.

a) Calculate the transition probability $P_{0\to n}(t, t_0)$ from the ground state $|0\rangle$ at $t_0 \to -\infty$ to an excited state $|n\rangle$ at $t \to +\infty$ in first order perturbation theory. What happens for $\tau \to 0$? 1pt(s)

Hint: Use $x = \sqrt{\frac{\hbar}{2m\omega}}\,\left(a^\dagger + a\right)$ to evaluate the matrix element.

b) The transition probability can also be calculated *exactly* using the following identity 1pt(s)

$$\hat{T}\, e^{-i\int_{t_0}^{t} \mathrm{d}t'\,\left(f(t')a + f^*(t')a^\dagger\right)} = e^{-i\int_{t_0}^{t} \mathrm{d}t' f(t') a}\, e^{-i\int_{t_0}^{t} \mathrm{d}t' f^*(t') a^\dagger}\, e^{\int_{t_0}^{t} \mathrm{d}t' f^*(t') \int_{t_0}^{t'} \mathrm{d}t'' f(t'')} \tag{3}$$

which is a generalization of the well-known relation for the displacement operator. Determine the time evolution for the initial state $|0\rangle$, and show that the transition probabilities $P_{0\to n}(t,t_0)$ for $t_0 \to -\infty$ and $t \to +\infty$ take the form

$$P_{0\to n} = \frac{K^{2n}}{n!} e^{-K^2} \quad \text{with} \quad K = \frac{eA}{2\sqrt{2m\omega\hbar}} e^{-\frac{\tau^2(\omega+\Omega)^2}{4}} \left(1 + e^{\tau^2\omega\Omega}\right) \tag{4}$$

and compare the result with a).

*c) Prove Eq. (3). **+1**pt(s)

Hint: Apply the same method as the proof of the relation $e^{A+B} = e^A\, e^B\, e^{-[A,B]/2}$ requires.

Problem 9.2: Spontaneous decay of the Hydrogen Atom [**Written** | 3 pt(s)]

ID: ex_spontaneous_decay_hydrogen_atom:aqt2223

Learning objective

In this exercise you will calculate which transitions in the Hydrogen atom can occur spontaneously. Basing on this you then will calculate the transition rates for electrons from the $n = 2$ manifold back to the ground state.

Consider an excited Hydrogen atom in the state $|n, l, m\rangle$ inside the electromagnetic vacuum. We try to find the possible transitions into a state $|n', l', m'\rangle$ by emitting a photon in the mode $|n_{k,\lambda} = 1\rangle$. Assuming the light-matter interaction is adiabatically turned on and off we can calculate the transition in first order perturbation theory as

$$\langle n', l', m'; n_{k,\lambda} = 1|\Psi(t)\rangle = \frac{1}{i\hbar} e^{-i\mathcal{E}_f(t-t_0)} \int_{t_0}^{t} dt_1\, \langle n', l', m'; n_{k,\lambda} = 1|\, H_{\text{int}}(t_1)\, |n, l, m; n_{k,\lambda} = 0\rangle\,. \tag{5}$$

In the lecture you showed that this results in the transition rate

$$\Gamma = \frac{d}{dt} |\,\langle n', l', m'; n_{k,\lambda} = 1|\Psi(t)\rangle\,|^2 = \frac{2\alpha\omega}{3} \frac{(\hbar\omega)^2}{mc^2 E_{\mathrm{R}}} |r_{ab}/a_{\mathrm{B}}|^2,$$

where E_{R} is the Rydberg energy, a_{B} the Bohr radius and ω is the frequency resonant to the transition $E_n \to E_{n'}$. Further, the dipole matrix element $\boldsymbol{r}_{ab}$ is given by

$$\boldsymbol{r}_{ab} = \langle n', l', m'|\, \boldsymbol{r}\, |n, l, m\rangle\,. \tag{6}$$

a) First rewrite $\boldsymbol{r} = (x, y, z)^T = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$. Then express this vector in terms of the spherical harmonics $Y_{l,m}(\theta,\phi)$. Use this to find the transitions which are allowed in first order. **1**pt(s)

b) Explicitly calculate the dipole matrix elements for the $n = 2 \to n = 1$ transition. **1**pt(s)

c) For an Hydrogen atom prepared in any of the $n = 2$ states, what are the average life times? **1**pt(s)

Problem 9.3: Photons as Fermions and violation of causality* [**Written** | 2 bonuspt(s)]

ID: ex_photons_fermions_violation_causality:aqt2223

Learning objective

In this problem we show that the causality is violated if we assume a fermionic statistics for photons.

a) If photons would be Fermions, the creation and annihilation operators $a^\dagger_{\mathbf{k},\lambda}$, $a_{\mathbf{k},\lambda}$ anticommute and satisfy the anti-commutation relation $\{a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$. Calculate the anti commutator for the electric field $\mathbf{E}$ **1**pt(s)

$$\{E_i(\mathbf{r},t), E_j(\mathbf{r}',t')\} = -\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\{A_i(\mathbf{r},t), A_j(\mathbf{r}',t')\}\,. \tag{7}$$

Focus on the equal time $\tau = t - t' = 0$ and show that the anti commutator is non vanishing. **Hint:** Follow the calculation of Problem 8.3.

b) Causality requires that observables commute with each other for space like separation. Show that the intensity E^2 does not commute at equal time arising from the above results. **Hint:** Calculate the commutator $\left[E_i^2(\mathbf{r}), E_j^2(\mathbf{r}')\right]$ and make use of anti-commutation relation for $E_i(\mathbf{r})$ and $E_j(\mathbf{r}')$. **1**pt(s)