Problem 3.1: Clebsch-Gordan coefficients and spin-orbit coupling

ID: ex_clebsch_gordan_coefficients_spin_orbit_coupling:aqt2223

Learning objective

In this problem you apply the angular momentum addition theorem. As an important use case, we consider the spin-orbit coupling in the hydrogen atom, which is the leading relativistic correction (see end of the course). In particular, you practice the construction of the new basis states for a fixed total angular momentum.

The spin-orbit coupling between the electron’s spin $S$ and the orbital angular momentum $L$ for a hydrogen atom is given by the Hamiltonian

$$H_{LS} = f(r) L \cdot S = f(r) \sum_{\alpha=x,y,z} L_\alpha \otimes S_\alpha,$$

where $f(r) = e^2 / 2m_e^2 c^2 r^3$. The spin-orbit coupling can be seen as a perturbation to the non-relativistic Hamiltonian $H_0 = P^2 / 2m - e^2 / r$ of the hydrogen atom.

a) Define the total angular momentum operator as

$$J = L + S = L \otimes 1 + 1 \otimes S$$

and show that $J^2$ and $J_z$ commute both with $H_0$ and $H_{LS}$.

b) Consider the subspace with orbital angular momentum $\ell$ and spin $s$. We can write the eigenstates $|j, m\rangle$ of $J^2$ and $J_z$ as linear combinations of $L_z$- and $S_z$-eigenstates $|\ell, m_\ell, m_s\rangle = |\ell, m_\ell\rangle \otimes |s, m_s\rangle$,

$$|j, m\rangle = \sum_{m_\ell, m_s} c(m_\ell, m_s; j, m) |m_\ell, m_s\rangle.$$

The coefficients $c$ are called **Clebsch-Gordan coefficients**. Due to their ubiquity in quantum physics there are comprehensive tables available, e.g.,


Use this table to write down the change of basis (3) in the subspace with $\ell = 1$ and $s = 1/2$ explicitly.

c) Derive the Clebsch-Gordan coefficients in b) by hand.

**Hint:** Start with the stretched state $|j = 3/2, m_j = 3/2\rangle$ and use the ladder operator $J_- = J_x - iJ_y$ which acts as

$$J_- |j, m\rangle = \hbar \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle.$$
Problem 3.2: Fine structure of the hydrogen atom

Learning objective
Here, we can apply the previously studied perturbation theory to derive the leading order energy corrections to the hydrogen atom due to spin-orbit coupling. This problem combines both topics, perturbation theory and addition of angular momenta, in one task.

a) Calculate the energy corrections to the $2p$ levels of the hydrogen atom (with the principal quantum number $n = 2$ and orbital angular momentum $l = 1$) arising from the spin-orbit coupling within the first order of perturbation theory.

Hint: Consider the coupling Hamiltonian $H_{LS}$ from Problem 3.1 as a perturbation to the unperturbed Hamiltonian $H_0$ of the hydrogen atom such that $H = H_0 + H_{LS}$. Show that the states with total angular momentum $J$ are eigenstates of $L \cdot S$. The first order energy corrections according to stationary perturbation theory can therefore in our case be calculated as

$$\Delta E^{(1)} = \langle \psi_{j,m} | H_{LS} | \psi_{j,m} \rangle.$$  (5)

For the radial part of the matrix elements use $\langle \psi_{2,1,m} | \frac{1}{r^3} | \psi_{2,1,m} \rangle = \frac{1}{24} a_0^3$, where $|\psi_{2,1,m}\rangle$ is an eigenfunction of $H_0$ and $a_0 = \frac{\hbar^2}{m e^2}$ is the Bohr radius.

*b) Derive the validity of equation (5) from the general perturbation theory.

Problem 3.3: System of three interacting spin-1/2 particles

Learning objective
In this exercise you will be going through an instructive example of a few particle spin system. The task is to show that using the theorem of addition of angular momentum, it is possible to analytically determine the ground state of three spins coupled antiferromagnetically; for a larger number of spins, this task is highly non-trivial and still poses unsolved problems in theoretical physics.

Let us consider a system composed of three spin-1/2 particles.

a) What is the dimension of the Hilbert space?

The total spin operator can be defined as $S = \sum_{i=1}^{3} S^{(i)}$ and its $z$ projection as $S_z = \sum_{i=1}^{3} S_z^{(i)}$. What are the eigenvalues and eigenstates of $S^2$ and $S_z$? Express the states in the spin-1/2 basis $|S_z^{(1)}, S_z^{(2)}, S_z^{(3)}\rangle$.

b) The Hamiltonian of the system is

$$H = J \sum_{i=1}^{3} S^{(i)} \cdot S^{(i+1)}, \quad J > 0.$$  

Here we assume a periodic system (for $i = 3$ take $i + 1 = 1$). Calculate the eigenstates and eigenenergies of this Hamiltonian.

Hint: Rewrite $H$ as a function of $S^2$ and $S^{(i)2}$. 