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Problem 2.1: Orbital Angular Momentum

[**Oral** | 4 pt(s)]

ID: ex_angular_momentum:aqt2223

Learning objective

In the following exercise we focus on the angular momentum operator. With its help we can describe rotations and introduce the concepts of scalar and vector operators.

Consider the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. In the lecture it was shown that $\mathbf{L}$ is the infinitesimal generator of rotations such that rotations around some axis $\mathbf{n}$ with $\mathbf{n}^2 = 1$ about some angle ω can be written as $U_{\boldsymbol{\omega}} = \exp\left(-i\omega\mathbf{L} \cdot \mathbf{n}/\hbar\right)$. If $U_{\boldsymbol{\omega}}$ is the operator performing a rotation around some axis $\boldsymbol{\omega} = \omega\mathbf{n}$ in the Hilbert space, i.e. $|\phi_{\boldsymbol{\omega}}\rangle = U_{\boldsymbol{\omega}}\,|\phi\rangle$; a *scalar* operator S transforms like

$$U_{\boldsymbol{\omega}}^\dagger\, S\, U_{\boldsymbol{\omega}} = S\,, \tag{1}$$

and a *vector* operator $\mathbf{X}$ transforms like

$$U_{\boldsymbol{\omega}}^\dagger\, \mathbf{X}\, U_{\boldsymbol{\omega}} = R_{\boldsymbol{\omega}}\, \mathbf{X}\,, \tag{2}$$

where $R_{\boldsymbol{\omega}}$ is the usual rotation matrix in three dimensions around some axis $\boldsymbol{\omega}$.

a) Show that for a scalar operator S, $[\mathbf{L}, S] = 0$. 1pt(s)

b) Show that for a vector operator $\mathbf{X}$ it is $[L_i, X_j] = i\hbar\varepsilon_{ijk}X_k$. 1pt(s)
Hint: Use the representation $(\mathcal{R}_{\boldsymbol{\omega}})_{ij} = [1-\cos(\omega)]\hat{\omega}_i\hat{\omega}_j + \cos(\omega)\,\delta_{ij} - \sin(\omega)\,\varepsilon_{ijk}\hat{\omega}_k$ for the rotation matrix and linearize (2) for small ω.

c) Using that $\mathbf{r}$ and $\mathbf{p}$ are vector operators, show that $\mathbf{L}$ is also a vector operator. 1pt(s)
Hint: Consider the components of $U_{\boldsymbol{\omega}}^\dagger\, \mathbf{r} \times \mathbf{p}\, U_{\boldsymbol{\omega}}$ and show that $U_{\boldsymbol{\omega}}^\dagger\, \mathbf{r} \times \mathbf{p}\, U_{\boldsymbol{\omega}} = U_{\boldsymbol{\omega}}^\dagger\, \mathbf{r}\, U_{\boldsymbol{\omega}} \times U_{\boldsymbol{\omega}}^\dagger\, \mathbf{p}\, U_{\boldsymbol{\omega}}$.

d) Show that $[\mathbf{L}, \mathbf{p} \cdot \mathbf{r}] = 0$ on the one hand by explicitely calculating the commutator and on the other hand by showing that $\mathbf{p} \cdot \mathbf{r}$ is a scalar operator. 1pt(s)

Problem 2.2: Pauli Matrices

[**Written** | 4 pt(s)]

ID: ex_pauli_matrices:aqt2223

Learning objective

The Pauli matrices are very important for the description of two-level systems. In this exercise you will derive some useful properties of the Pauli matrices.

The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

a) Prove that the Pauli matrices fulfill the following commutation relation: **1**pt(s)

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{4}$$

b) Show that **1**pt(s)

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k. \tag{5}$$

Use this relation to prove that

$$(\mathbf{r} \cdot \boldsymbol{\sigma})^2 = |\mathbf{r}|^2\mathbb{1}, \tag{6}$$

with $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$.

c) For spin 1/2 the spin operator $\mathbf{S} = (S_x, S_y, S_z)^T$ can be written in terms of Pauli matrices $\mathbf{S} = \hbar/2\boldsymbol{\sigma}$. Show that the representation of the rotation takes the form **1**pt(s)

$$U_{\theta\hat{\mathbf{n}}} = \exp\left(-\frac{i}{\hbar}\theta\, \mathbf{S} \cdot \hat{\mathbf{n}}\right) = \mathbb{1}\cos\frac{\theta}{2} - i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\sin\frac{\theta}{2}, \tag{7}$$

where $\hat{\mathbf{n}}$ is the rotation axis and θ the rotation angle.
Consider the state $|\uparrow\rangle = (1, 0)^T$. What is the expectation value of the spin operator after rotation by $\pi/2$ and π around the y axis?

d) Show that the spin operator $\mathbf{S}$ transforms like a vector under rotation. **1**pt(s)
Hint: Refer to subtask b of problem 2.1.

Problem 2.3: Hydrogen Atom – Lowest States

[**Oral** | 2 pt(s)]

ID: ex_hydrogen_atom_lowest_states:aqt2223

Learning objective

In this exercise, we will derive the wave functions of the Hydrogen atom for the ground state and first excited states, and determine the size of the atoms where electrons occupy these lower states.

Consider the wave functions of the Hydrogen atom as derived in the Lecture, $\psi_{n,\ell,m}$.

a) Write explicitly the $1s,\ 2s$, and $2p_z$ wave functions, which correspond to the set of quantum numbers $\{n, \ell, m\} = \{1, 0, 0\},\ \{2, 0, 0\}$, and $\{2, 1, 0\}$, respectively. **1**pt(s)

b) Determine the expectation value of the radial components r, r^2 and $1/r$ in the ground state (i.e. the $1s$ state) and the $2p_z$ state. Deduce the size of the atom in each of these states. **1**pt(s)