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Problem 15.1: Addition of angular momentum [2 pt(s)]

ID: ex_addition_angular_momentum_repetition:aqt2223

A hydrogen atom has an electron with spin $\boldsymbol{S}$ and $S = 1/2$ and orbital angular momentum $\boldsymbol{L}$, as well as a nucleus with nuclear spin $\boldsymbol{I}$ and $I = 1/2$.

a) Consider the system in its electronic ground state with orbital angular momentum $l = 0$ and principal quantum number $n = 1$. What is the ground state degeneracy? What values can the total angular momentum $\boldsymbol{F} = \boldsymbol{L} + \boldsymbol{S} + \boldsymbol{I}$ assume? **1** pt(s)

b) Now consider the system to be in a state with orbital angular momentum $l = 1$. What are the possible values for the total angular momentum and what are the corresponding degeneracies? **1** pt(s)

Problem 15.2: Time-dependent perturbation theory [4 pt(s)]

ID: ex_time_dependent_perturbation_theory_repetition:aqt2223

We analyze a one-dimensional harmonic oscillator with mass m and frequency ω in a time-dependent electric field. The Hamiltonian has the form $H = H_0 + H'(t)$ with $H_0 = p^2/2m + m\omega^2 x^2/2$ and the perturbation $H'(t) = xD(t)$. The time-dependence of the electric field has the form

$$D(t) = \frac{A}{\tau\sqrt{\pi}} e^{-(t/\tau)^2} \tag{1}$$

with the constants $A > 0$ and $\tau > 0$. Determine the transition probability $P_{0\to n}(\infty)$ from the ground state $|0\rangle$ to an exited state $|n\rangle$ in first order perturbation theory, i.e. the transition probability for large times $t \to \infty$. What happens for $\tau \to 0$?

Problem 15.3: Quantization of the radiation field [9 pt(s)]

ID: ex_quantization_radiation_field_repetition:aqt2223

a) Write down the Hamiltonian of the quantized radiation field (photons). What is the Hilbert space called? Give a full basis of the Hilbert space. **1** pt(s)

b) What is the field operator $\hat{\boldsymbol{B}}(\boldsymbol{r}, t)$ of the magnetic field? Explain the different terms. **1** pt(s)

Now we consider an atom with the ground state $|g\rangle$ and exited state $|e\rangle$ with energies E_g and E_e. We want to investigate the decay of the exited state due to interactions with the radiation field. The interactions of both states with the radiation field can be written as

$$H_{\text{int}} = -\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{B}} = -(|g\rangle\langle e| + |e\rangle\langle g|)\boldsymbol{d} \cdot \hat{\boldsymbol{E}} \tag{2}$$

with the dipole moment $\boldsymbol{d} = \langle g|\hat{\boldsymbol{d}}|e\rangle = \langle e|\hat{\boldsymbol{d}}|g\rangle$ between both states.

c) Write down Fermis golden rule. Initially the atom is in the exited state and the radiation field is in the vacuum state. What is the final state after the decay? **1** pt(s)

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d) Calculate the matrix element in Fermis golden rule and determine the rate of decay into a specific mode. **2**pt(s)

e) Calculate the density of states for free photons. **2**pt(s)

f) Now calculate the rate of decay of the exited state, where we are not interested in which mode the photon is emitted. What role does the polarization play? **2**pt(s)

Problem 15.4: Second quantization [7 pt(s)]

ID: ex_second_quantization_repetition:aqt2223

Let $\psi^\dagger(\boldsymbol{x})$ be a fermionic field operator. The field operator describes a spinless particle and only consists of one component. Furthermore we define an operator

$$B^\dagger(\boldsymbol{R}) = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\phi(\boldsymbol{r})\psi^\dagger(\boldsymbol{R}+\boldsymbol{r}/2)\psi^\dagger(\boldsymbol{R}-\boldsymbol{r}/2) \tag{3}$$

where the wave function $\phi(\boldsymbol{r})$ describes a bound state. Thus for $|\boldsymbol{r}| > a_0$ we can set the wave function $\phi(\boldsymbol{r}) = 0$, where the characteristic length a_0 describes the size of the bound state. Furthermore the wave function is normalized, i.e. $\int \mathrm{d}\boldsymbol{r}|\phi(\boldsymbol{r})|^2 = 1$.

a) What relations do the field operators $\psi^\dagger(\boldsymbol{x})$ and $\psi(\boldsymbol{y})$ fulfill? **2**pt(s)

b) Which of the following relations can be fulfilled by $\phi(\boldsymbol{r})$ (justify your answer)? **2**pt(s)

$$\phi(\boldsymbol{r}) = \phi(-\boldsymbol{r}) \quad \phi(\boldsymbol{r}) = -\phi(-\boldsymbol{r}) \quad \phi(\boldsymbol{r}) = \phi(\boldsymbol{r})^* \tag{4}$$

c) Calculate the commutator $[B(\boldsymbol{R}'), B^\dagger(\boldsymbol{R})]$. Explain, why for large distances $|\boldsymbol{R}-\boldsymbol{R}'| \gg a_0$ the operator $B^\dagger(\boldsymbol{R})$ describes the field operator of a bosonic particle. **2**pt(s)

d) Give an example from nature where two fermions bind to form a bosonic particle. **1**pt(s)

Problem 15.5: Dirac equation [4 pt(s)]

ID: ex_dirac_equation_repetition:aqt2223

a) Write down the Dirac equation in its covariant form with help of the γ-matrices. How many γ-matrices are there? How many components does the Dirac-spinor ψ have? **2**pt(s)

b) Under a Lorentz-transformation Λ a contravariant vector x^μ transforms as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. How does the differential operator of a contravariant vector $\partial/\partial x^\mu$ transform? **1**pt(s)

c) The spinor ψ transforms via $\psi'(x') = S(\Lambda)\psi(x)$. What are the requirements that S and the γ-matrices need to fulfill, such that the Dirac equation is lorentz invariant (with derivation/justification)? **1**pt(s)