Prof. Dr. Hans-Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

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Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Fort. Vielteilchentheorie / Adv. Quantum Theory"***:

- The `C@MPUS-IDs` of this course are 049350000 (Fortgeschrittene Vielteilchentheorie) and 049160000 (Advanced Quantum Theory).
- You can find detailed information on lecture and tutorials on the website of our institute:
  `https://www.itp3.uni-stuttgart.de/teaching/aqt2223/`
- You can also find detailed information on lecture and tutorials on `ILIAS`:
  `https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_3022297.html`
- **Written** problems have to be handed in and will be corrected by the tutors. You must earn at least **80 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **2** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Foundations of Quantum Mechanics [**Oral** | 6 pt(s)]

ID: ex_foundations_of_quantum_mechanics:aqt2223

Learning objective

This problem recapitulates foundational concepts of quantum mechanics and its formalism. It is based on *Student understanding of quantum mechanics*, C. Singh, Am. J. Phys. **69**, 885 (2001); `http://dx.doi.org/10.1119/1.1365404` to prevent and/or eradicate common misconceptions. A thorough and intuitive understanding of all these concepts is essential.

Try to take the following short test without using any reference material.

We refer to a generic observable Q and its corresponding quantum mechanical operator $\hat{Q}$. For all of the questions, the Hamiltonian and operators $\hat{Q}$ do not depend on time explicitly.

a) The eigenvalue equation for an operator $\hat{Q}$ is given by $\hat{Q}\,|\psi_i\rangle = \lambda_i\,|\psi_i\rangle$, where $i = 1, \ldots, N$. Write an expression for $\langle\phi|\hat{Q}|\phi\rangle$, where $|\phi\rangle$ is a general state, in terms of the amplitudes $\langle\phi|\psi_i\rangle$. **1**pt(s)

b) If you make measurements of a physical observable Q on a system in immediate succession, do you expect the outcome to be the same every time? Justify your answer. **1**pt(s)

c) If you make measurements of a physical observable Q on an ensemble of identically prepared systems which are not in an eigenstate of $\hat{Q}$, do you expect the outcome to be the same every time? Justify your answer. **1**pt(s)

d) A particle is in a one-dimensional harmonic oscillator potential. Under what conditions will the expectation value of an operator $\hat{Q}$ depend on time if **1**pt(s)

  i. the particle is initially in a momentum eigenstate?

  ii. the particle is initially in an energy eigenstate?

  Justify your answer in both cases.

e) Questions (i)-(ix) refer to the following system: An electron is in a uniform magnetic field B which is pointing in the z-direction. The Hamiltonian for the spin-degree of freedom for this system is given by $\hat{H} = -\gamma B \hat{S}_z$, where γ is the gyromagnetic ratio and $\hat{S}_z$ is the z-component of the spin angular momentum operator. **2**pt(s)

  Notation: $\hat{S}_z \left|\uparrow\right\rangle = \hbar/2 \left|\uparrow\right\rangle, \hat{S}_z \left|\downarrow\right\rangle = -\hbar/2 \left|\downarrow\right\rangle$.
  For reference, the unnormalized eigenstates of $\hat{S}_x$ and $\hat{S}_y$ are given by

$$\hat{S}_x(\left|\uparrow\right\rangle \pm \left|\downarrow\right\rangle) = \pm\hbar/2(\left|\uparrow\right\rangle \pm \left|\downarrow\right\rangle) \tag{1}$$

$$\hat{S}_y(\left|\uparrow\right\rangle \pm i\left|\downarrow\right\rangle) = \pm\hbar/2(\left|\uparrow\right\rangle \pm i\left|\downarrow\right\rangle)\,. \tag{2}$$

  i. If you measure S_z of a state $\left|\chi\right\rangle = (\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)/\sqrt{2}$, what are the possible results, and what are their respective probabilities?

  ii. If the result of the first measurement of S_z was $\hbar/2$, and you immediately measure S_z again, what are the possible results, and what are their respective probabilities?

  iii. If the result of the first measurement of S_z was $\hbar/2$, and you immediately measure S_x, what are the possible results, and what are their respective probabilities?

  iv. What is the expectation value $\langle \hat{S}_z \rangle$ of the state $\left|\chi\right\rangle = (\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)/\sqrt{2}$?

  v. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_y$ depend on time? Justify your answer.

  vi. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_z$ depend on time? Justify your answer.

  vii. If the electron is initially in an eigenstate of $\hat{S}_z$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.

  viii. If the electron is initially in an eigenstate of $\hat{S}_z$, does the expectation value of $\hat{S}_z$ depend on time? Justify your answer.

  ix. If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.

Problem 1.2: Commutators

[**Written** | 5 pt(s)]

ID: ex_commutators_2:aqt2223

> **Learning objective**
>
> Here we recapitulate some important commutation relations with focus on position, (linear) momentum, and angular momentum operators. As commutators are ubiquitous in quantum mechanics, manipulating them should become second nature to you.

a) Show that the following identity holds for commutators of products **1**pt(s)

$$[A, BC] = B[A, C] + [A, B]C \,. \tag{3}$$

b) Let $g(x)$ and $f(p)$ be analytical functions of the position and momentum operators, respectively. Show that **1**pt(s)

$$[p, g(x)] = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} g(x) \,, \tag{4}$$

$$[x, f(p)] = +i\hbar \frac{\mathrm{d}}{\mathrm{d}p} f(p) \,. \tag{5}$$

c) Calculate the following commutators for canonical position and momentum operators x and p **1**pt(s)

$$\left[x, p^2\right] , \qquad \left[x^2, p^2\right] , \qquad \left[xp, p^2\right] . \tag{6}$$

d) Using just the canonical commutation relations of x and p, calculate $[L_\alpha, L_\beta]$, where $L_\alpha = \varepsilon_{\alpha ij} x_i p_j$ is the α-component of the angular momentum operator. **1**pt(s)

e) Now calculate $[L_\alpha, \boldsymbol{L}^2]$ and $[L_\alpha, p^2]$. **1**pt(s)

Problem 1.3: Creation and Annihilation Operators

[**Oral** | 5 pt(s)]

ID: ex_creation_and_annihilation_operators:aqt2223

> **Learning objective**
>
> This problem recapitulates the paradigmatic harmonic oscillator and unveils the underlying algebraic structure by the introduction of creation and annihilation operators. The concept of creation and annihilation operators will be used a lot during this advanced quantum mechanics course; being familiar with them is therefore an important prerequisite for the lecture.

The Hamiltonian of a one-dimensional harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \,. \tag{7}$$

a) As usual, we introduce creation and annihilation operators $a^\dagger$ and a via **1**pt(s)

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p \right) , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} p \right) . \tag{8}$$

Calculate the commutation relations for these operators.

b) Express the Hamiltonian in terms of the number operator $n = a^\dagger a$. **1**pt(s)

c) Let $|n\rangle$ be the eigenvector of $a^\dagger a$ with eigenvalue n. How can $|n\rangle$ be expressed in terms of the creation operator $a^\dagger$ and the ground state $|0\rangle$ (no calculation needed)? **1**pt(s)

d) What is the ground state wavefunction $\psi_0(x) = \langle x|0\rangle$? Use the coordinate representation of the creation operator to calculate the excited state $\psi_1(x) = \langle x|1\rangle$. **1**pt(s)

e) Show that the creation operator $a^\dagger$ has no (right-)eigenvector. Does the same argument hold for the annihilation operator a? **1**pt(s)